Integration by parts and convergence in distribution norms in the CLT

Vlad Bally, L. Caramellino, G. Poly

University de Marne-la-Vallée, Tor Vergata and Rennes

2024 *Le Mans*

CLT $X_i \in R^d, i \in N, \quad i.i.d. \quad with \quad E(X_i) = 0, \quad E(X_i^k X_i^p) = \delta_{k,p}$

Denote

$$S_n(X) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

Then

$$E(\phi(S_n(X)) \to E(\phi(N)) \quad with \quad N \sim N(0, I).$$

Test functions

$$\phi \in C_b \quad \to \quad \text{weak convergence}$$
$$\phi \text{ measurable bounded} \quad \to \quad \text{convergence in total variation}$$

THEOREM (Prohorov) Convergence in total variation is equivalent to : there exists m such that

$$Y_1 = X_1 + ... + X_m \sim f(x)dx + \nu(dx).$$

Remark

$$Z = Y_1 + Y_2 \sim g(x)dx + \mu(dx) \quad g \in C(R)$$

Consequence

$$g \geq \varepsilon \mathbf{1}_{B_r(x_0)} \geq \varphi_r(x - x_0) \quad \varphi_r \in C^\infty$$

Splitting

$$Z \sim \xi \times V + (1 - \xi)U$$

With : ξ, V, U independent with laws

$$P(V \in dx) = \frac{1}{c_r}\varphi_r(x - x_0)dx \quad \varphi_r \simeq \mathbf{1}_{B_r(0)}$$
$$P(\xi = 1) = \varepsilon c_r, \quad P(\xi = 0) = 1 - \varepsilon c_r$$
$$P(U \in dx) = \frac{1}{1 - \varepsilon c_r}(P(Z \in dx) - \varepsilon\varphi_r(x - x_0)dx)$$

Now on we take

$$Z_i \sim \xi_i \times V_i + (1 - \xi_i)U_i \quad i \in N$$

Strategy : we use an "abstract Malliavin calculus" based on $V_1, V_2, ..., V_n, ...$ in order to prove Prohorov's Theorem.

Integration by parts Suppose that

$$V \sim \varphi(x)dx$$

Then

$$\begin{aligned} E(f'(V)) &= \int_R f'(x)\varphi(x)dx = -\int_R f(x)\varphi'(x)dx \\ &= -\int_R f(x)\frac{\varphi'(x)}{\varphi(x)} \times \varphi(x)dx \\ &= -E(f(V)\frac{\varphi'(V)}{\varphi(V)}) \\ &= E(f(V)H(V)) \end{aligned}$$

Conclusion

$$E(f'(V)) = E(f(V)H(V)) \quad with \quad H(V)) = -\frac{\varphi'(V)}{\varphi(V)} = -\partial \ln \varphi(V).$$

Come back on the end

MAIN RESULTS

Convergence in total variation.

$$\left|E(f(S_n(X)) - E(f(N))\right| \leq \frac{C}{\sqrt{n}} \left\| f \right\|_\infty$$

and moreover

Convergence in (almost) distribution norms : let $|\alpha| = q$.

$$\left|E(\partial^\alpha f(S_n(X)) - E(\partial^\alpha f(N))\right| \leq \frac{C}{\sqrt{n}} \left\| f \right\|_\infty + e^{-cn} \left\| f \right\|_{q,\infty}$$

with

$$\left\| f \right\|_{q,\infty} = \sum_{|\alpha| \leq q} \left\| \partial_\alpha f \right\|_\infty .$$

Corollary

$$\gamma_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{1/2}} e^{-\frac{|x|^2}{2\varepsilon}} \quad f_\varepsilon = f * \gamma_\varepsilon$$

Then, with $\varepsilon = \varepsilon_n = e^{-cn}$

$$|E(\partial_\alpha f_{\varepsilon_n}(S_n(X)) - E(\partial^\alpha f(N))| \leq \frac{C}{\sqrt{n}} \|f\|_\infty$$

Densities : Stronger hypothesis :

Corollary If $Z_i \sim p(x)dx, \quad p \in W^{1,1}$ $(\nabla p \in L^1)$ Then $S_n \sim p_{S_n}(x)dx$ and

$$\left\| \partial^\alpha p_{S_n(X)} - \partial^\alpha \gamma \right\|_\infty \leq \frac{C}{\sqrt{n}} \quad \forall\, |\alpha| \leq n \quad \gamma(x) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{|x|^2}{2}}$$

Remark

$$Z_1 + ... + Z_n \sim (p * ... * p)(x)dx \quad (p * ... * p) \in W^{n,1}$$

Extensions

A. $Z_i, i \in N$ are **not identically distributed**

B **Local theorems (Edgeworth expansions)**

$$\left| E(\partial_\alpha f(S_n) - \int \partial_\alpha f(x) (\sum_{i=0}^{k} \frac{1}{n^{i/2}} \theta_i(x)) \gamma(x) dx \right| \leq \frac{C}{n^{(k+1)/2}} \|f\|_\infty + e^{-cn} \|f\|_{q,\infty}$$

Expected number of roots for trigonometric polynomials

$$P_n(t,Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (Y_k^1 \cos \frac{kt}{n} + Y_k^2 \sin \frac{kt}{n})$$
$$N_n(Y) = card\{t \in (0, n\pi) : P_n(t,Y) = 0\}.$$

Invarience principle

$$\lim_n \frac{1}{n} E(N_n(Y)) = \lim_n \frac{1}{n} E(N_n(G)) = \frac{1}{\sqrt{3}}$$

Sckech of the proof

Kac-Rice lemma let $f : [0,a] \to R$ differentiable such that

$$\inf_{t \le a} (|f(t)| + \left|f'(t)\right|) > 0.$$

Then

$$N_a(f) = \lim_{\delta \to 0} \int_0^a \left|f'(t)\right| 1_{\{|f(t)| \le \delta\}} \frac{dt}{2\delta}.$$

We use the CLT for $S_n = (S_n^1, S_n^2) \in R^2$ with

$$S_n^1(t,Y) = P_n(t,Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (Y_k^1 \cos\frac{kt}{n} + Y_k^2 \sin\frac{kt}{n})$$

$$S_n^2(t,Y) = \partial_t P_n(t,Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{k}{n}(Y_k^2 \cos\frac{kt}{n} - Y_k^1 \sin\frac{kt}{n})$$

We take

$$\Phi_\delta(x^1, x^2) \simeq |x_2| \times 1_{(-\infty,0)}^{(\delta)}(x^1) \quad \rightarrow \quad \partial_1 \Phi_\delta(x_1, x_2) \approx x_2 \frac{1}{2\delta} 1_{\{|x_1| \le \delta\}}$$

K-R lemma

$$N_n(Y) = \lim_{\delta \to 0} \int_0^{n\pi} \partial_1 \Phi_\delta(S_n^1(t,Y), S_n^2(t,Y)) dt$$

with

$$\partial_1 \Phi_\delta(S_n^1(t,Y), S_n^2(t,Y)) \approx S_n^2(t,Y) \frac{1}{2\delta} 1_{\{|S_n^1(t,Y)| \le \delta\}}$$

CLT (invarience principle)

$$\lim_n E(\partial_1 \Phi_\delta(S_n^1(t,Y), S_n^2(t,Y))) = \lim_n E(\partial_1 \Phi_\delta(S_n^1(t,G), S_n^2(t,G))$$

Varience

$$\lim_n \frac{1}{n} Var(N_n(G)) = c(G) = 0.56.$$

We prove (non-universality) :

$$\lim_n \frac{1}{n} Var(N_n(Y)) = c(G) + 30(E(Y_1^4) - 3).$$

Basic quantity

$$\frac{1}{n} \int_0^{\pi n} dt \int_0^{\pi n} ds \quad \partial_1 \Phi_\delta(S_n(t,Y)) \partial_1 \Phi(S_n(s,Y))$$

We use CLT for

$$\overline{S}_n(t,s,Y) = (S_n(t,Y), S_n(s,Y)) \in R^4$$

with the Edgorth developpement of order 4.

Difficulties

1 Degeneracy on the dyagonal :

$$s \to t \quad \Rightarrow \quad S_n(s, Y) \to S_n(t, Y)$$

2 Magnitude order

$$\frac{1}{n} \int_0^{\pi n} dt \int_0^{\pi n} ds \quad \sim \quad \frac{1}{n} \times n^2 = n$$

So we need

$$n \times \left| \overline{S}_n(t, s, Y) - \overline{S}_n(t, s, G) \right| \to 0.$$

Estimates of order $n^{-3/2}$ in the CLT. Use Edgworth at order 3.

Integration by parts (abstract Malliavin)

$$F = f(\xi_i V_i + (1-\xi_i)U_i, i = 1, ..., n)$$

$$P(V \in dx) = \frac{1}{c_r}\varphi_r(x - x_0)dx \quad \varphi_r \simeq 1_{B_r(0)}$$

Derivatives

$$D_k F = \xi_k \frac{\partial}{V_k} f(\xi_i V_i + (1-\xi_i)U_i, i = 1, ..., n)$$

Duality

$$E(\langle DF, DG\rangle) = \sum_{k=1}^{n} E(D_k F D_k G) = E(FLG)$$

with

$$LG = -\sum_{k=1}^{N} D_k D_k G + D_k G \partial_v \ln \varphi_r(V_k - x_0)$$

Proof Standard Integration by parts :

$$
\begin{aligned}
&E(D_kFD_kG) \\
&= E(\xi_k \frac{\partial}{V_k} f(\xi_k V_k + (1-\xi_k)U_k, \ldots) \frac{\partial}{V_k} g(\xi_k V_k + (1-\xi_k)U_k, \ldots)) \\
&= -E(\xi_k \int \partial_k f(\xi_k v_k \ldots) \partial_k g(\xi_k v_k, \ldots) \varphi_r(v_k - x_0) dv_k \\
&= -E(\xi_k \int f \times (\partial_k^2 g \times \varphi_k + \partial_k g \varphi_k') dv_k \\
&= -E(\xi_k \int f \times (\partial_k^2 g + \partial_k g \frac{\varphi_k'}{\varphi_k}) \times \varphi_r(v_k - x_0) dv_k \\
&= -E(F(D_kD_kG + D_kG \partial_v \ln \varphi_r(V_k - x_0))
\end{aligned}
$$

Covarience matrix : $F=(F_1,...,F_d)$

$$\sigma_F^{i,j}=\left\langle DF_i,DF_j\right\rangle, \quad i,j=1,...,d.$$

Difficulty

$$P(\det\sigma_F=0)\geq P(\xi_1=...=\xi_n=0)>0.$$

Hoffding's inequality

$$P(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i<\frac{c_r}{2})\leq e^{-an}.$$

IbyP

$$E(\partial^\alpha f(F)G)=E(f(F)H_\alpha(F,G))$$

with

$$\|H_\alpha(F,G)\|_p\leq E(\frac{1}{(\det\sigma_F)^{p'}})\times(\|F\|_{q,p}+\|G\|_{q,p}).$$

Regularization lemma $|\alpha| = m$

$$|E(\partial^\alpha f(F)) - E(\partial^\alpha (f * \gamma_\varepsilon)(F)| \leq C \|f\|_{m,\infty} P^{1/2}(\det \sigma_F \leq \eta) + \frac{\varepsilon}{\eta^{2m}} \|f\|_\infty$$
$$\leq C \|f\|_{m,\infty} e^{-an} + \frac{\varepsilon}{\eta^{2m}} \|f\|_\infty$$

Here γ_ε is a **super kernel**

In particular

$$|E(f(F)) - E(f * \gamma_\varepsilon)(F)| \leq (P^{1/2}(\det \sigma_F \leq \eta) + \frac{\varepsilon}{\eta^{2m}}) \|f\|_\infty$$
$$\leq C(e^{-an} + \frac{\varepsilon}{\eta^{2m}}) \|f\|_\infty$$

Proof $\gamma_\varepsilon \geq 0$ is a super kernel if $\int \gamma_\varepsilon(x) = 1$ and

$$\int x^p \gamma_\varepsilon(x) dx = 0 \quad \forall p \geq 1.$$

It follows that, for every m

$$\begin{aligned}(f*\gamma_\varepsilon)(F)-f(F) &= \int \gamma_\varepsilon(y)f(F+y)dy - f(F)\\ &= \int \gamma_\varepsilon(y)(f(F+y)-f(F))dy\\ &= \sum_{i=1}^{m}\int \gamma_\varepsilon(y)\times\frac{y^i}{i!}dy\times f^{(i)}(F)\\ &+\int \gamma_\varepsilon(y)R_m(f)(y)dy\end{aligned}$$

with

$$R_m(f)(y)=\frac{y^{m+1}}{(m+1)!}\int_0^1 d\lambda f^{(m+1)}(\lambda F+(1-\lambda)y)$$

And we use IbP in order to handel

$$E(\int \gamma_\varepsilon(y)R_m(f)(y)dy)$$

Estimates

$$
\begin{aligned}
&|E(f(F)) - E(f(G)| \\
&\leq |E(f(F)) - E((f * \gamma_\varepsilon)(F)| + |E(f(G)) - E((f * \gamma_\varepsilon)(G)| \\
&+ |E(f * \gamma_\varepsilon(F)) - E((f * \gamma_\varepsilon)(G)| \\
&\leq C(e^{-an} + \frac{\varepsilon}{\eta^{2m}}) \|f\|_\infty + \frac{1}{\varepsilon} E|F - G| \times \|f\|_\infty
\end{aligned}
$$

1. Bally, V., Caramellino, L., Poly, G. (2018) Convergence in distribution norms in the CLT for non identical distributed random variables. Electronic Journal of Probability, 23, no. 45, 1-51.
2. Bally, V., Caramellino, L., Poly, G. (2017) On the non universality for the variance of the number of real roots of random trigonometric polynomials. Preprint arXiv :1711.0336 43 pages. November 2017 Probability Theory and Related Fields
3. Bally, V., Caramellino, L., Poly, G. (2019) : Regularization lemmas and convergence in total variation. ArXiv :1907.12328, 2019. Electronic Journal of Probability