Space-time white noises in a nonlinear expectation space

Xiaojun Ji and Shige Peng, May 21, 2024

Dedicated to 30 Anniversary
of Le Mans Mathematical Laboratory
also to Jean-Pierre Lepeltier

Outline

Background

- Kolmogorov's foundation of probability theory: $(\Omega, \mathcal{F}, P)$
  - Wiener probability space: $\Omega = C([0, \infty)), \mathcal{F} = \mathcal{B}(\Omega)$
  - Brownian motion: $B_t(\omega) = \omega_t, t \geq 0.$
- Knight (1921): Knightian uncertainty
- Choquet (1953): Choquet expectation, Capacity theory
- Peng (1997): g-expectation, conditional g-expectation

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- Knight (1921): Knightian uncertainty
- Choquet (1953): Choquet expectation, Capacity theory
- Peng (1997): g-expectation, conditional g-expectation
- Peng (2004): Nonlinear (sublinear) expectation theory $(\Omega, \mathcal{H}, \mathbb{E})$
  - $\mathbb{E}[X] = \sup_{P\in\mathcal{P}} E_P[X] = \sup_{P\in\mathcal{P}} \int_\Omega X dP$

- Example: Nonlinear g-expectation
- Consider BSDE on $(\Omega, L_P^2(\mathcal{F}_T))$

$$-dY_t^\xi = g(Z_t)dt - Z_t^\xi dW_t, \quad Y_T^\xi = \xi \in L_P^2(\mathcal{F}_T)$$

The g-expectation and g-conditional expectation:

$$\hat{\mathbb{E}}^g[\xi|\mathcal{F}_t] := Y_t^\xi, \quad : L_P^2(\mathcal{F}_T) \mapsto L_P^2(\mathcal{F}_t), \quad 0 \le t \le T.$$

$\hat{\mathbb{E}}^g[\xi] = \hat{\mathbb{E}}^g[\xi|\mathcal{F}_0] := Y_0^\xi,.$

Nonlinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$

- Ω is a given set
- $\mathcal{H}$ is a linear space of real-valued functions on Ω such that

  $X_1, \cdots, X_n \in \mathcal{H}$, then $\varphi(X_1, \cdots, X_n) \in \mathcal{H} \implies$ for each $\varphi \in C_{Lip}(\mathbb{R}^n)$[1].
- $\mathcal{H}$ is considered as the space of random variables.

[1] $C_{Lip}(\mathbb{R}^n)$ denotes the set of all Lipschitz functions on $\mathbb{R}^n$

Nonlinear expectation space

Definition 1

A **nonlinear expectation** is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for each $X, Y \in \mathcal{H}$,

(i) **Monotonicity:** $X \geq Y$ implies $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;

(ii) **Constant preserving:** $\hat{\mathbb{E}}[c] = c$ for $c \in \mathbb{R}$;

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(iii) **Sub-additivity:** $\hat{\mathbb{E}}[X+Y]\leq\hat{\mathbb{E}}[X]+\hat{\mathbb{E}}[Y]$;

(iv) **Positive homogeneity:** $\hat{\mathbb{E}}[\lambda X]=\lambda\hat{\mathbb{E}}[X]$ for $\lambda>0$.

The triple $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ is called a nonlinear (sublinear) expectation space.

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- If the inequality in (iii) becomes equality, $\hat{\mathbb{E}}$ reduces to a linear expectation and $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ reduces to a linear expectation space.

Robust representation theorem

(v) **Regularity:** If $\{X_i\}_{i=1}^{\infty} \subset \mathcal{H}$ satisfies that $X_i(\omega) \downarrow 0$ as $i \to \infty$, for each $\omega \in \Omega$, then

$$\lim_{i\to\infty} \hat{\mathbb{E}}[X_i] = 0.$$

Theorem 2

Let $\hat{\mathbb{E}}$ be a sublinear expectation on $(\Omega, \mathcal{H})$ satisfying the regularity condition. Then there exists a weakly compact set $\mathcal{P}$ of probability measures on $(\Omega, \sigma(\mathcal{H}))$, such that

$$\hat{\mathbb{E}}[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi], \text{ for each } \xi \in \mathcal{H}.$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

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- Capacity:

$$c(A) = \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Distribution

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a nonlinear (resp. sublinear) expectation space. For each d-dimensional random vector $X \in \mathcal{H}^d$, define $\mathbb{F}_X : C_{Lip}(\mathbb{R}^d) \to \mathbb{R}$ by

$$\mathbb{F}_X[\varphi] := \hat{\mathbb{E}}[\varphi(X)], \ \forall \varphi \in C_{Lip}(\mathbb{R}^d). \tag{1.1}$$

$\mathbb{F}_X$ is called the **distribution** of X. $(\mathbb{R}^d, C_{Lip}(\mathbb{R}^d), \mathbb{F}_X)$ forms a sublinear expectation space.

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Definition 3

Two d-dimensional random vectors on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$, respectively, are called **identically distributed**, denoted by $X_1 \overset{d}{=} X_2$, if $\mathbb{F}_{X_1} = \mathbb{F}_{X_2}$, i.e.,

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \ \forall \varphi \in C_{Lip}(\mathbb{R}^d). \tag{1.2}$$

Independence

Definition 4

A d-dimensional random vector Y is said to be **independent** from an n-dimensional random vector X, denoted by $Y \perp\!\!\!\perp X$, if for each test function $\varphi \in C_{Lip}(\mathbb{R}^{n+d})$,

$$\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}]. \tag{1.3}$$

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- "$Y \perp\!\!\!\perp X$" $\not\Rightarrow$ "$X \perp\!\!\!\perp Y$" (See Peng (2010), Hu & Li (2014))
- Let $\bar{X}$ and X be two d-dimensional random vectors on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. $\bar{X}$ is called an **independent copy** of X if $\bar{X} \overset{d}{=} X$ and $\bar{X} \perp\!\!\!\perp X$.

G-normal distribution

Definition 5

A d-dimensional random vector X on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G**-normally distributed** if

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X, \text{ for } a, b \geq 0,$$

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- $\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0$.
- For $d = 1$, $X \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$, where $\underline{\sigma}^2 := -\hat{\mathbb{E}}[-X^2]$, $\overline{\sigma}^2 := \hat{\mathbb{E}}[X^2]$.

$$G_X(a) := \frac{1}{2}\hat{\mathbb{E}}[aX^2] = \frac{1}{2}\overline{\sigma}^2 a^+ - \frac{1}{2}\underline{\sigma}^2 a^-, \ \forall a \in \mathbb{R}.$$

- G_X is called the **generating function** of X.

Relation to the G-heat equation

Let G be the generating function of the G-normally distributed random variable X. For each $\varphi \in C_{Lip}(\mathbb{R}^d)$, define

$$u(t,x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)], \ (t,x) \in [0,\infty) \times \mathbb{R}^d. \tag{1.4}$$

Proposition 6

u is the unique viscosity solution of the G-heat equation

$$\partial_t u - G(D_x^2 u) = 0, \ \ u|_{t=0} = \varphi(x). \tag{1.5}$$

Generating function

For a d-dimensional G-normally distributed random vector X, the generating function $G = G_X : \mathbb{S}(d) \mapsto \mathbb{R}$ is defined by

$$G_X(Q) := \frac{1}{2}\hat{\mathbb{E}}[\langle QX, X\rangle], \quad Q \in \mathbb{S}(d).$$

where $\mathbb{S}(d)$ denotes the collection of all $d \times d$ symmetric matrices.

- G is a sublinear and continuous function monotone in $Q \in \mathbb{S}(d)$.
- There exists a bounded and closed set $\Upsilon \subset \mathbb{S}(d)$ such that

$$G(Q) = \frac{1}{2}\sup_{\nu\in\Upsilon} \mathrm{tr}[\nu Q], \; Q \in \mathbb{S}(d).$$

- A d-dimensional G-normally distributed random vector is denoted by $X \sim \mathcal{N}(0, \Upsilon)$.

Generating function

Proposition 7

Let ξ be a d-dimensional G-normally distributed random vector characterized by its generating function

$$G_\xi(Q) := \frac{1}{2}\hat{\mathbb{E}}[\langle Q\xi, \xi\rangle], \; Q \in \mathbb{S}(d).$$

Then, for any matrix $K \in \mathbb{R}^{m\times d}$, $K\xi$ is also an m-dimensional G-normally distributed random vector. Its corresponding generating function is

$$G_{K\xi}(Q) = \frac{1}{2}\hat{\mathbb{E}}[\langle K^T QK\xi, \xi\rangle], \; Q \in \mathbb{S}(m).$$

G-Brownian motion

Definition 8

A d-dimensional process $(B_t)_{t\geq 0}$ with $B_t \in \mathcal{H}^d$ for each $t \geq 0$ is called a **G-Brownian motion** if the following properties are satisfied:

(1) $B_0 = 0$;

(2) For each $t, s \geq 0$, $B_{t+s} - B_t \sim \mathcal{N}(0, s\Upsilon)$;

(3) For each $t, s \geq 0$, $B_{t+s} - B_t \perp\!\!\!\perp (B_{t_1}, \ldots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.

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- $(B_{t_1}, \ldots, B_{t_n})$ is not G-normally distributed.
- G-Brownian motion is not a G-Gaussian process.

G-Gaussian random field

Let Γ be a parameter set. Denote the family of all sets of finite indices by

$$\mathcal{J}_\Gamma := \{\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) : \ \forall n \in \mathbb{N}, \ \gamma_1, \cdots, \gamma_n \in \Gamma, \ \gamma_i \neq \gamma_j \text{ for } i \neq j\}.$$

Definition 9

A d-dimensional **random field** on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is a family of random variables $W = (W_\gamma)_{\gamma \in \Gamma}$ such that $W_\gamma \in \mathcal{H}^d$ for each $\gamma \in \Gamma$.

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Definition 10

A d-dimensional random field $(W_\gamma)_{\gamma \in \Gamma}$ on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G**-Gaussian random field** if for each $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$, the $(d \times n)$-dimensional random vector $W_{\underline{\gamma}} = (W_{\gamma_1}, \cdots, W_{\gamma_n})$ is G-normally distributed.

- G-Brownian motion $\not\Rightarrow$ G-Gaussian random field

For each $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$, we define

$$G_{W_{\underline{\gamma}}}(Q) = \frac{1}{2}\hat{\mathbb{E}}[\langle QW_{\underline{\gamma}}, W_{\underline{\gamma}}\rangle], \quad Q \in \mathbb{S}(n \times d),$$

Then $(G_{W_{\underline{\gamma}}})_{\underline{\gamma} \in \mathcal{J}_\Gamma}$ constitutes a family of monotone sublinear and continuous functions satisfying the properties of consistency:

(1) Compatibility: For any $(\gamma_1, \cdots, \gamma_n, \gamma_{n+1}) \in \mathcal{J}_\Gamma$ and $Q \in \mathbb{S}(n \times d)$,

$$G_{W_{\gamma_1}, \cdots, W_{\gamma_n}, W_{\gamma_{n+1}}}(\bar{Q}) = G_{W_{\gamma_1}, \cdots, W_{\gamma_n}}(Q), \tag{2.1}$$

where $\bar{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{S}((n+1) \times d)$;

(2) Symmetry: For any permutation π of $\{1, \cdots, n\}$ and $Q \in \mathbb{S}(n \times d)$,

$$G_{W_{\gamma_{\pi(1)}}, \cdots, W_{\gamma_{\pi(n)}}}(Q) = G_{W_{\gamma_1}, \cdots, W_{\gamma_n}}(\pi^{-1}(Q)), \tag{2.2}$$

where the mapping $\pi^{-1} : \mathbb{S}(n \times d) \mapsto \mathbb{S}(n \times d)$ is defined by

$$\left(\pi^{-1}(Q)\right)_{ij} = (q_{\pi^{-1}(i)\pi^{-1}(j)}), \ \ i, j = 1, \cdots, n \times d.$$

Existence of G-Gaussian random fields

Theorem 11

Let $(G_{\underline{\gamma}})_{\underline{\gamma}\in\mathcal{J}_\Gamma}$ be a family of monotonic and sublinear functions satisfying the compatibility condition (2.1) and symmetry condition (2.2). Then there exists a d-dimensional G-Gaussian random field $(W_\gamma)_{\gamma\in\Gamma}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ such that for each $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$, $W_{\underline{\gamma}} = (W_{\gamma_1}, \cdots, W_{\gamma_n})$ is G-normally distributed and

$$G_{W_{\underline{\gamma}}}(Q) = \frac{1}{2}\hat{\mathbb{E}}[\langle QW_{\underline{\gamma}}, W_{\underline{\gamma}}\rangle] = G_{\underline{\gamma}}(Q), \quad \textit{for any } Q \in \mathbb{S}(n \times d).$$

Existence of G-Gaussian random fields

Theorem 12

If there exists another Gaussian random field $(\bar{W}_\gamma)_{\gamma\in\Gamma}$, with the same index set Γ, defined on a sublinear expectation space $(\bar{\Omega}, \bar{\mathcal{H}}, \bar{\mathbb{E}})$ such that for each $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$, $\bar{W}_{\underline{\gamma}}$ is G-normally distributed with the same generating function, namely,

$$\frac{1}{2}\bar{\mathbb{E}}[\langle Q\bar{W}_{\underline{\gamma}}, \bar{W}_{\underline{\gamma}}\rangle] = G_{\underline{\gamma}}(Q) \quad \text{for any } Q \in \mathbb{S}(n \times d).$$

Then we have $W \overset{d}{=} \bar{W}$.

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Remark

If $\Gamma = \mathbb{R}^+$, $W = (W_\gamma)_{\gamma\in\Gamma}$ becomes a G-Gaussian process which has been studied in Peng (2011).

Spatial G-white noise

Let $\Gamma = \mathcal{B}_0(\mathbb{R}^d) = \{A \in \mathcal{B}(\mathbb{R}^d), \lambda_A < \infty\}$, where λ_A denotes the Lebesgue measure of $A \in \mathcal{B}(\mathbb{R}^d)$.

Definition 13

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space. A 1-dimensional G-Gaussian random field $\mathbb{W} = (\mathbb{W}_A)_{A \in \Gamma}$ is called a 1-dimensional **G-white noise** if

(1) For all $A \in \Gamma$, $\hat{\mathbb{E}}[\mathbb{W}_A^2] = \overline{\sigma}^2 \lambda_A$, $-\hat{\mathbb{E}}[-\mathbb{W}_A^2] = \underline{\sigma}^2 \lambda_A$;

(2) For each $A_1, A_2 \in \Gamma$, $A_1 \cap A_2 = \varnothing$, we have

$$\hat{\mathbb{E}}[\mathbb{W}_{A_1} \mathbb{W}_{A_2}] = \hat{\mathbb{E}}[-\mathbb{W}_{A_1} \mathbb{W}_{A_2}] = 0, \tag{2.3}$$

$$\hat{\mathbb{E}}[(\mathbb{W}_{A_1 \cup A_2} - \mathbb{W}_{A_1} - \mathbb{W}_{A_2})^2] = 0, \tag{2.4}$$

where $0 \leq \underline{\sigma}^2 \leq \overline{\sigma}^2$ are any given numbers.

Existence of G-white noise

Set

$$G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-), \ a \in \mathbb{R}. \tag{2.5}$$

For each $\underline{\gamma} = (A_1, \cdots, A_n)$, $A_j \in \Gamma = \mathcal{B}_0(\mathbb{R}^d)$, define a sublinear and monotone function $G_{\underline{\gamma}}(\cdot) : \mathbb{S}(n) \mapsto \mathbb{R}$ as follows:

$$G_{A_1,\cdots,A_n}(Q) = G\left(\sum_{i,j=1}^{n} q_{ij} \lambda_{A_i \cap A_j}\right), \ Q = (q_{ij})_{i,j=1}^{n} \in \mathbb{S}(n). \tag{2.6}$$

- $(G_{\underline{\gamma}})_{\underline{\gamma} \in \mathcal{J}_\Gamma}$ satisfies the compatibility condition (2.1) and symmetry condition (2.2).

Existence of G-white noise

Theorem 14

For each given numbers $0 \le \underline{\sigma}^2 \le \overline{\sigma}^2$, *let the family of generating functions* $(G_{\underline{\gamma}})_{\underline{\gamma} \in \mathcal{J}_\Gamma}$ *be defined as in* (2.6). *Then there exists a* 1*-dimensional spatial* G*-white noise* $(\mathbb{W}_\gamma)_{\gamma \in \Gamma}$ *on a sublinear expectation space* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *such that, for each* $\underline{\gamma} = (A_1, \cdots, A_n) \in \mathcal{J}_\Gamma$, $Q = (q_{ij})_{i,j=1}^n \in \mathbb{S}(n)$,

$$G_{\mathbb{W}_{\underline{\gamma}}}(Q) = \frac{1}{2}\hat{\mathbb{E}}[\langle Q\mathbb{W}_{\underline{\gamma}}, \mathbb{W}_{\underline{\gamma}}\rangle] = G\Big(\sum_{i,j=1}^{n} q_{ij}\lambda_{A_i \cap A_j}\Big).$$

- Denote by $\mathbb{L}_G^2(\mathbb{W})$ the completion of $\mathcal{H}$ under the Banach norm $\|\cdot\|_2 = (\hat{\mathbb{E}}[|\cdot|^2])^{1/2}$. Then $(\Omega, \mathbb{L}_G^2(\mathbb{W}), \hat{\mathbb{E}})$ forms a complete sublinear expectation space.

Invariance under rotation and translation

Proposition 15

For each $p \in \mathbb{R}^d$ *and* $O \in \mathbb{O}(d) := \{O \in \mathbb{R}^{d\times d} : O^T = O^{-1}\}$, *we set*

$$T_{p,O}(A) = \{Ox + p : x \in A\}, \text{ for } A \in \mathcal{B}_0(\mathbb{R}^d).$$

Then, for each $A_1, \cdots, A_n \in \mathcal{B}_0(\mathbb{R}^d)$, *we have*

$$(\mathbb{W}_{A_1}, \cdots, \mathbb{W}_{A_n}) \stackrel{d}{=} (\mathbb{W}_{T_{p,O}(A_1)}, \cdots, \mathbb{W}_{T_{p,O}(A_n)}).$$

Namely, the finite-dimensional distributions of $\mathbb{W}$ *are invariant under rotations and translations.*

Stochastic calculus w.r.t. G-white noise

For any simple function

$$f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x), \ \forall n \in \mathbb{N}, a_1, \cdots, a_n \in \mathbb{R}, A_1, \cdots, A_n \in \Gamma,$$

define the stochastic integral w.r.t. the spatial G-white noise as follows:

$$\int_{\mathbb{R}^d} f(x)\mathbb{W}(dx) = \sum_{i=1}^{n} a_i \int_{\mathbb{R}^d} \mathbf{1}_{A_i}(x)\mathbb{W}(dx) = \sum_{i=1}^{n} a_i \mathbb{W}_{A_i}.$$

Denote $L^2(\mathbb{R}^d) = \{f : \|f\|_{L^2}^2 = \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty.\}$

Stochastic calculus w.r.t. G-white noise

Lemma 16

If $f:\mathbb{R}^d\to\mathbb{R}$ is a simple function, then

$$\hat{\mathbb{E}}\left[\left|\int_{\mathbb{R}^d} f(x)\mathbb{W}(dx)\right|^2\right]\le\overline{\sigma}^2\|f\|_{L^2}^2.$$

- The stochastic integral can be continuously extended to the whole domain of $L^2(\mathbb{R}^d)$.

Theorem 17

$\{\int_{\mathbb{R}^d} f(x)\mathbb{W}(dx): f\in L^2(\mathbb{R}^d)\}$ *is a G-Gaussian random field.*

Example 18

Let $\{\mathbb{W}_A, A \in \mathcal{B}_0(\mathbb{R})\}$ be a 1-dimensional G-white noise. Define $\mathbb{B}_t = \mathbb{W}([0,t]), t \in \mathbb{R}_+$, then

$$\hat{\mathbb{E}}[\mathbb{B}_t\mathbb{B}_s] = \overline{\sigma}^2\lambda_{[0,t]\cap[0,s]} = \overline{\sigma}^2(s \wedge t).$$

Unlike the classical case, $\mathbb{B}_t$ is no longer a G-Brownian motion, although $\mathbb{B}_t \overset{d}{=} N(\{0\} \times [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$ for each $t \geq 0$.

Space-time G-white noise

Set $\Gamma = \{[s,t) \times A : 0 \le s \le t < \infty,\ A \in \mathcal{B}_0(\mathbb{R}^d)\}$.

Definition 19

A random field $\{\mathbf{W}([s,t) \times A)\}_{([s,t)\times A)\in\Gamma}$ on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a 1-dimensional **space-time G-white noise** if it satisfies the following conditions:

(i) For each fixed $[s,t)$, $\{\mathbf{W}([s,t) \times A)\}_{A\in\mathcal{B}_0(\mathbb{R}^d)}$ is a 1-dimensional spatial G-white noise that has the same family of finite-dimensional distributions as $(\sqrt{t-s}\mathbb{W}_A)_{A\in\mathcal{B}_0(\mathbb{R}^d)}$;

(ii) For any $r \le s \le t$, $A \in \mathcal{B}_0(\mathbb{R}^d)$,

$$\mathbf{W}([r,s) \times A) + \mathbf{W}([s,t) \times A) = \mathbf{W}([r,t) \times A);$$

(iii) For any $t_i \le s \le t$ and $A_i \in \mathcal{B}_0(\mathbb{R}^d)$, $i = 1, \cdots, n$,

$$\mathbf{W}([s,t) \times A) \perp\!\!\!\perp (\mathbf{W}([s_1,t_1) \times A_1), \cdots, \mathbf{W}([s_n,t_n) \times A_n)),$$

where $(\mathbb{W}_A)_{A\in\mathcal{B}_0(\mathbb{R}^d)}$ is a 1-dimensional spatial G-white noise.

Remark

It is important to mention that $\{\mathbf{W}([s,t)\times A), 0 \le s \le t < \infty, A \in \mathcal{B}_0(\mathbb{R}^d)\}$ is no longer a G-Gaussian random field.

Existence of G-white noise

Set

$$\Omega = \Big\{\omega \in (\mathbb{R})^{\Gamma} : \omega([r,t)\times A) = \omega([r,s)\times A) + \omega([s,t)\times A), \\ \forall r \le s \le t,\ A \in \mathcal{B}_0(\mathbb{R}^d)\Big\},$$

and for each $\omega \in \Omega$, define the canonical process $(\mathbf{W}_\gamma)_{\gamma\in\Gamma}$ by

$$\mathbf{W}([s,t)\times A)(\omega) = \omega([s,t)\times A),\ \forall 0 \le s \le t < \infty,\ A \in \mathcal{B}_0(\mathbb{R}^d).$$

Set $\mathcal{F}_T = \sigma\{\mathbf{W}([s,t)\times A), 0 \le s \le t \le T, A \in \mathcal{B}_0(\mathbb{R}^d)\}$, $\mathcal{F} = \bigvee_{T\ge 0} \mathcal{F}_T$, and

$$L_{ip}(\mathcal{F}_T) = \{\varphi(\mathbf{W}([s_1,t_1)\times A_1),\cdots,\mathbf{W}([s_n,t_n)\times A_n)), \forall n \in \mathbb{N},\ s_i \le t_i \le T, \\ i = 1,\cdots,n,\ \ A_1,\cdots,A_n \in \mathcal{B}_0(\mathbb{R}^d), \varphi \in C_{Lip}(\mathbb{R}^n)\},$$

$$L_{ip}(\mathcal{F}) = \bigcup_{n=1}^{\infty} L_{ip}(\mathcal{F}_n).$$

For $X \in L_{ip}(\mathcal{F})$ with the form

$$X = \varphi(\mathbf{W}([0,t_1)\times A_1),\cdots,\mathbf{W}([0,t_1)\times A_m),\cdots,\mathbf{W}([t_{n-1},t_n)\times A_1),\cdots,$$
$$\mathbf{W}([t_{n-1},t_n)\times A_m)),$$

where $0 < t_1 < \cdots < t_n < \infty$, $\{A_1,\cdots,A_m\} \subset \mathcal{B}_0(\mathbb{R}^d)$ are mutually disjoint, and $\varphi \in C_{Lip}(\mathbb{R}^{n\times m})$. Define

$$\hat{\mathbb{E}}[X] := \tilde{\mathbb{E}}[\varphi(\sqrt{t_1}\xi_1^{(1)},\cdots,\sqrt{t_1}\xi_1^{(m)},\cdots,\sqrt{t_n - t_{n-1}}\xi_n^{(1)},\cdots,\sqrt{t_n - t_{n-1}}\xi_n^{(m)})],$$

where $\{\xi_1,\cdots,\xi_n\}$, $\xi_j = (\xi_j^{(1)},\cdots,\xi_j^{(m)})$, $1 \le j \le n$, are i. i. d. G-normally distributed random vectors on a sublinear expectation space $(\tilde{\Omega},\tilde{\mathcal{H}},\tilde{\mathbb{E}})$.

The conditional expectation of X under $\mathcal{F}_t$, $t_j \leq t < t_{j+1}$, is defined by

$$\begin{aligned}\hat{\mathbb{E}}[X|\mathcal{F}_t] :=& \hat{\mathbb{E}}[\varphi(\mathbf{W}([0,t_1)\times A_1),\cdots,\mathbf{W}([0,t_1)\times A_m),\cdots,\\ &\mathbf{W}([t_{n-1},t_n)\times A_1),\cdots,\mathbf{W}([t_{n-1},t_n)\times A_m))|\mathcal{F}_t]\\ =& \psi(\mathbf{W}([0,t_1)\times A_1),\cdots,\mathbf{W}([0,t_1)\times A_m),\cdots,\\ &\mathbf{W}([t_{j-1},t_j)\times A_1),\cdots,\mathbf{W}([t_{j-1},t_j)\times A_m)),\end{aligned}$$

where

$$\begin{aligned}\psi(x_{11},\cdots,x_{jm}) = \tilde{\mathbb{E}}[\varphi(&x_{11},\cdots,x_{jm},\sqrt{t_{j+1}-t_j}\,\xi_{j+1}^{(1)},\cdots,\sqrt{t_{j+1}-t_j}\,\xi_{j+1}^{(m)},\\ &\cdots,\sqrt{t_n-t_{n-1}}\,\xi_n^{(1)},\cdots,\sqrt{t_n-t_{n-1}}\,\xi_n^{(m)})].\end{aligned}$$

- The canonical process $(\mathbf{W}_\gamma)_{\gamma\in\Gamma}$ is a space-time white noise on $(\Omega, L_{ip}(\mathcal{F}), \hat{\mathbb{E}}, (\hat{\mathbb{E}}[\cdot|\mathcal{F}_t])_{t\geq 0})$.
- For each $p \geq 1$, denote by $\mathbf{L}_G^p(\mathbf{W}_{[0,T]})$ (resp., $\mathbf{L}_G^p(\mathbf{W})$) the completion of $L_{ip}(\mathcal{F}_T)$ (resp., $L_{ip}(\mathcal{F})$) under the form $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$.

Proposition 20

For any $X, Y \in \mathbf{L}_G^p(\mathbf{W})$, $\eta \in \mathbf{L}_G^p(\mathbf{W}_{[0,t]})$, *we have*

(i) $\hat{\mathbb{E}}[X|\mathcal{F}_t] \geq \hat{\mathbb{E}}[Y|\mathcal{F}_t]$ *for* $X \geq Y$.

(ii) $\hat{\mathbb{E}}[\eta|\mathcal{F}_t] = \eta$.

(iii) $\hat{\mathbb{E}}[X+Y|\mathcal{F}_t] \leq \hat{\mathbb{E}}[X|\mathcal{F}_t] + \hat{\mathbb{E}}[Y|\mathcal{F}_t]$.

(iv) $\hat{\mathbb{E}}[\eta X|\mathcal{F}_t] = \eta^+\hat{\mathbb{E}}[X|\mathcal{F}_t] + \eta^-\hat{\mathbb{E}}[-X|\mathcal{F}_t]$ *if* η *is bounded.*

(v) $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\mathcal{F}_t]|\mathcal{F}_s] = \hat{\mathbb{E}}[X|\mathcal{F}_{t\wedge s}]$ *for* $s \geq 0$.

Stochastic integral w.r.t. space-time G-white noise

- Let $\mathbf{M}^{2,0}([0,T]\times\mathbb{R}^d)$ be the collection of simple random fields with the form:

$$f(s,x;\omega)=\sum_{i=0}^{n-1}\sum_{j=1}^{m}X_{ij}(\omega)\mathbf{1}_{A_j}(x)\mathbf{1}_{[t_i,t_{i+1})}(s), \tag{3.1}$$

  where $X_{ij}\in\mathbf{L}_G^2(\mathbf{W}_{[0,t_i]})$, $i=0,\cdots,n-1$, $j=1,\cdots,m$, $0=t_0<\cdots<t_n=T$, and $\{A_j\}_{j=1}^m\subset\mathcal{B}_0(\mathbb{R}^d)$ is a mutually disjoint sequence.

- Bochner's integral of f:

$$\int_{\mathbb{R}^d}\int_0^T f(s,x)dsdx:=\sum_{i=0}^{n-1}\sum_{j=1}^{m}X_{ij}(t_{i+1}-t_i)\lambda_{A_j}. \tag{3.2}$$

Stochastic integral w.r.t. space-time G-white noise

The stochastic integral w.r.t. the space-time G-white noise $\mathbf{W}$ can be defined as follows:

$$\int_0^T \int_{\mathbb{R}^d} f(s,x)\mathbf{W}(ds,dx) := \sum_{i=0}^{n-1}\sum_{j=1}^{m} X_{ij}\mathbf{W}([t_i,t_{i+1})\times A_j). \tag{3.3}$$

- $\mathbf{M}^{2,0}([0,T]\times\mathbb{R}^d) \mapsto \mathbf{L}_G^2(\mathbf{W}_{[0,T]})$

Stochastic integral w.r.t. space-time G-white noise

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- $\mathbf{M}^{2,0}([0,T]\times\mathbb{R}^d) \mapsto \mathbf{L}_G^2(\mathbf{W}_{[0,T]})$

Lemma 21

For any simple random field $f \in \mathbf{M}^{2,0}([0,T]\times\mathbb{R}^d)$,

$$\hat{\mathbb{E}}\left[\int_0^T \int_{\mathbb{R}^d} f(s,x)\mathbf{W}(ds,dx)\right] = 0, \tag{3.4}$$

$$\hat{\mathbb{E}}\left[\left|\int_0^T \int_{\mathbb{R}^d} f(s,x)\mathbf{W}(ds,dx)\right|^2\right] \le \overline{\sigma}^2 \hat{\mathbb{E}}\left[\int_0^T \int_{\mathbb{R}^d} |f(s,x)|^2 dsdx\right]. \tag{3.5}$$

Stochastic integral w.r.t. space-time G-white noise

- Denote by $\mathbf{M}_G^2([0,T]\times\mathbb{R}^d)$ the completion of $\mathbf{M}^{2,0}([0,T]\times\mathbb{R}^d)$ under the norm $\|\cdot\|_{\mathbf{M}^2} := \left(\hat{\mathbb{E}}[\int_0^T\int_{\mathbb{R}^d}|\cdot|^2 dx ds]\right)^{1/2}$.
- The stochastic integral can be continuously extended to $\mathbf{M}_G^2([0,T]\times\mathbb{R}^d)$.

Proposition 22

For each $f,g\in\mathbf{M}_G^2([0,T]\times\mathbb{R}^d)$, $0\le s\le r\le t\le T$, we have

(i) $\int_s^t\int_{\mathbb{R}^d} f(u,x)\mathbf{W}(du,dx) =$
$\int_s^r\int_{\mathbb{R}^d} f(u,x)\mathbf{W}(du,dx) + \int_r^t\int_{\mathbb{R}^d} f(u,x)\mathbf{W}(du,dx)$.

(ii) *If $\alpha\in\mathbf{L}_G^1(\mathbf{W}_{[0,s]})$ is bounded,*
$\int_s^t\int_{\mathbb{R}^d}(\alpha f(r,x)+g(r,x))\mathbf{W}(dr,dx)$
$=\alpha\int_s^t\int_{\mathbb{R}^d} f(r,x)\mathbf{W}(dr,dx) + \int_s^t\int_{\mathbb{R}^d} g(r,x)\mathbf{W}(dr,dx)$.

(iii) $\hat{\mathbb{E}}[\int_s^T\int_{\mathbb{R}^d} f(r,x)\mathbf{W}(dr,dx)|\mathcal{F}_s] = 0$.

G-stochastic heat equation

Consider the stochastic heat equation driven by the multiplicative space-time G-white noise:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + b(u) + a(u)\dot{\mathbf{W}}(t,x), \ 0 < t \leq T, 0 \leq x \leq L, \\ \frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,L) = 0, \ 0 < t \leq T, \\ u(0,x) = u_0(x), \ 0 \leq x \leq L, \end{cases} \tag{4.1}$$

where u_0 is a bounded function and $a(x), b(x) \in C_{Lip}(\mathbb{R})$.

- $\dot{\mathbf{W}}(t,x)$ is the generalized mixed derivative of space-time G-white noise $\mathbf{W}$.

G-stochastic heat equation

- $\Gamma = [0,T] \times \mathbb{R}$
- Space-time G-white noise $\mathbf{W} = \{\mathbf{W}(t,x) : t \in [0,T], x \in \mathbb{R}\}$:

$$\mathbf{W}(t,x) := \mathbf{W}([0,t) \times [0 \wedge x, 0 \vee x]), \text{ for } t \in [0,T], x \in \mathbb{R}.$$

- The generalized mixed derivative $\dot{\mathbf{W}}(t,x)$ is defined by the test function $\phi \in C_c^\infty(\mathbb{R}^2)$ as follows:

$$\int_0^T \int_{\mathbb{R}} \dot{\mathbf{W}}(t,x)\phi(t,x)dxdt := \int_0^T \int_{\mathbb{R}} \mathbf{W}(t,x) \frac{\partial^2 \phi(t,x)}{\partial t \partial x} dxdt.$$

G-stochastic heat equation

- $\Gamma = [0,T] \times \mathbb{R}$
- Space-time G-white noise $\mathbf{W} = \{\mathbf{W}(t,x) : t \in [0,T], x \in \mathbb{R}\}$:

$$\mathbf{W}(t,x) := \mathbf{W}([0,t) \times [0 \wedge x, 0 \vee x]), \text{ for } t \in [0,T], x \in \mathbb{R}.$$

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Proposition 23

For each $\phi \in C_c^\infty(\mathbb{R}^2)$, we have

$$\int_0^T \int_{\mathbb{R}} \dot{\mathbf{W}}(t,x)\phi(t,x)dxdt = \int_0^T \int_{\mathbb{R}} \phi(t,x)\mathbf{W}(dt,dx). \tag{4.2}$$

G-stochastic heat equation

Definition 24

A spatio-temporal random field $\{u(t,x) : (t,x) \in [0,T] \times [0,L]\}$ is said to be a mild solution of the nonlinear G-stochastic heat equation (4.1) if it satisfies the following conditions:

(i) $(u(t,x))_{0<t\leq T, 0\leq x\leq L} \in \mathbf{S}^2_G([0,T]\times[0,L])$;

(ii) For $0 < t \leq T$, $0 \leq x \leq L$, in $\mathbf{L}^2_G(\mathcal{F}_t)$,

$$u(t,x) = \int_0^L u_0(y) g(t,x,y) dy + \int_0^t \int_0^L g(t-s,x,y) b(u(s,y)) dy ds + \int_0^t \int_0^L g(t-s,x,y) a(u(s,y)) \mathbf{W}(ds,dy).$$

- Here $g(t,x,y)$ denotes the Green's function for the linear heat equation.
- $\mathbf{S}^2_G([0,T]\times[0,L])$ is the completion of $\mathbf{M}^{2,0}([0,T]\times[0,L])$ under the norm $\|u\|_{\mathbf{S}^2} = \sup_{0\leq t\leq T} \sup_{0\leq x\leq L} (\hat{\mathbb{E}}[|u(t,x)|^2])^{\frac{1}{2}}$.

G-stochastic heat equation

Theorem 25

Let $u_0(x)$ be bounded and $a(x), b(x)$ be Lipschitz functions. Then nonlinear stochastic heat equation (4.1) driven by the multiplicative space-time G-white noise has a unique mild solution $\{u(t,x) : (t,x) \in [0,T] \times [0,L]\}$.

G-stochastic heat equation

Theorem 25

Let $u_0(x)$ be bounded and $a(x), b(x)$ be Lipschitz functions. Then nonlinear stochastic heat equation (4.1) driven by the multiplicative space-time G-white noise has a unique mild solution $\{u(t,x) : (t,x) \in [0,T] \times [0,L]\}$.

Remark

The mild solution is also a weak solution of the G-stochastic heat equation.

Thank you