On smooth change-point location estimation for Poisson processes and Skorokhod topologies

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Université de Lille

30 years of LMM

Le Mans — May 21, 2024

Recalls: $X = \big(X(t),\ 0 \le t \le T\big)$ is an (inhomogeneous) Poisson process of intensity function $\lambda(\cdot)$ if $X(0) = 0$ and the increments of X on disjoint intervals are independent Poisson random variables:

$$\mathbf{P}\Big\{X(t) - X(s) = k\Big\} = \frac{\left(\int_s^t \lambda(t)\,\mathrm{d}t\right)^k}{k!}\,\exp\left\{-\int_s^t \lambda(t)\,\mathrm{d}t\right\}.$$

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The process $X = \big(X(t),\ 0 \le t \le T\big)$ is a counting process. The (random) set $\big\{t_1, t_2, \cdots, t_{X(T)}\big\}$ of its jump times is a point process.

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The stochastic integral with respect to X is

$$\int_0^T f(t)\,\mathrm{d}X(t) = \sum_{i=1}^{X(T)} f(t_i).$$

Observation: $X^{(n)} = (X_1, \ldots, X_n)$, where $X_j = \bigl(X_j(t),\ 0 \leq t \leq \tau\bigr)$, $j = 1, \ldots, n$, are independent Poisson processes of intensity function $\lambda_\theta(\cdot)$ with some unknown parameter θ.

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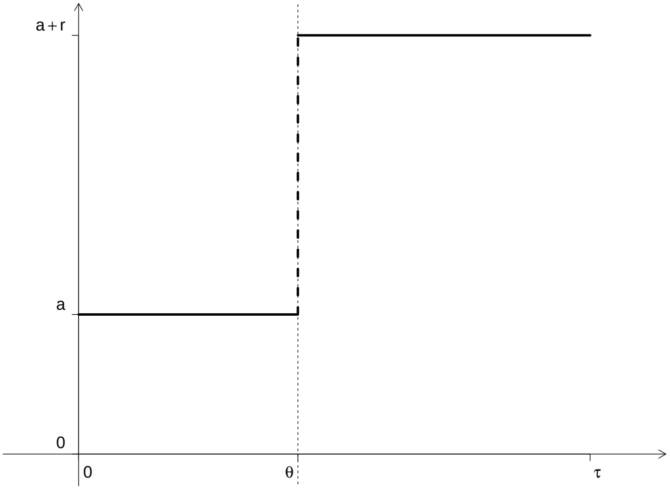

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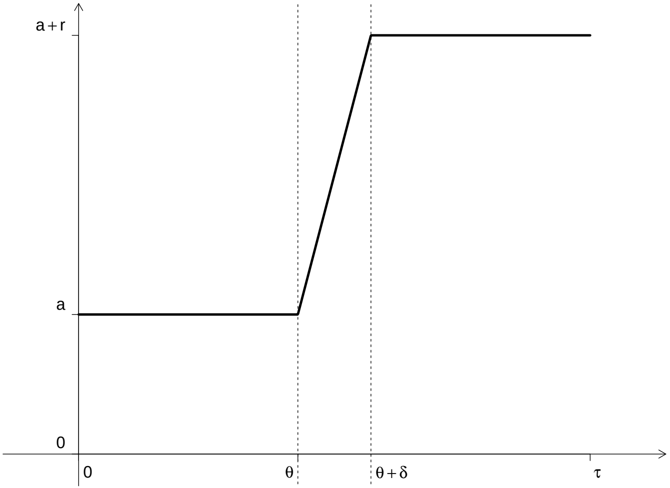

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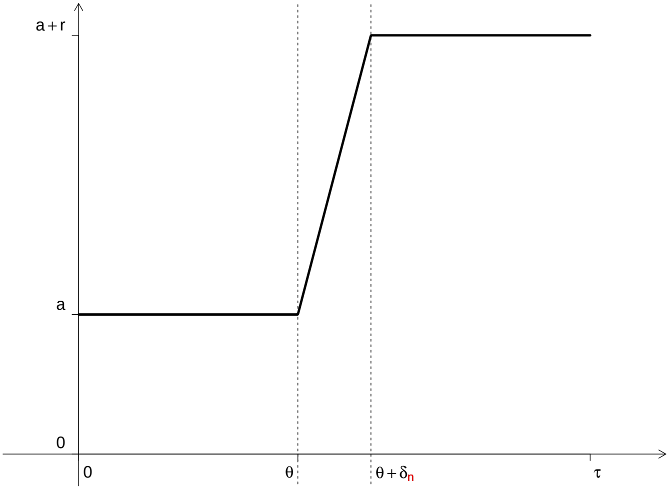

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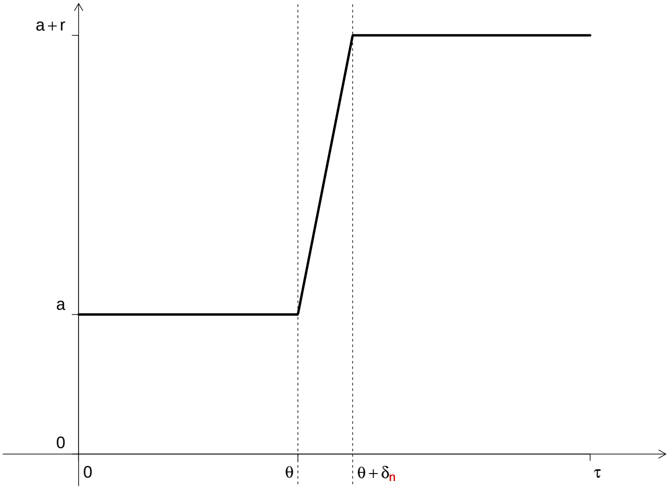

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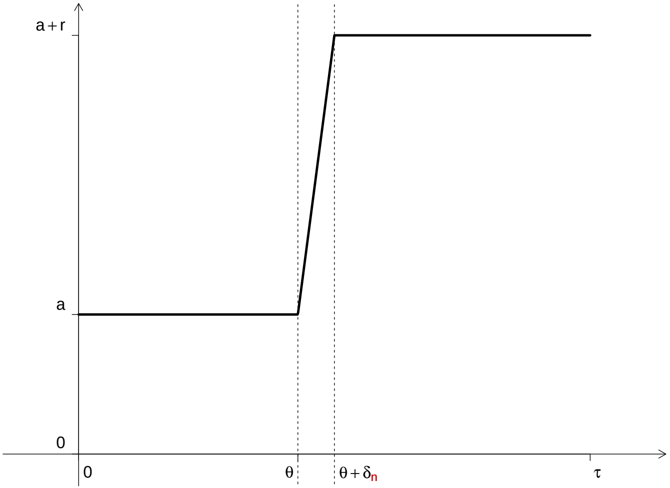

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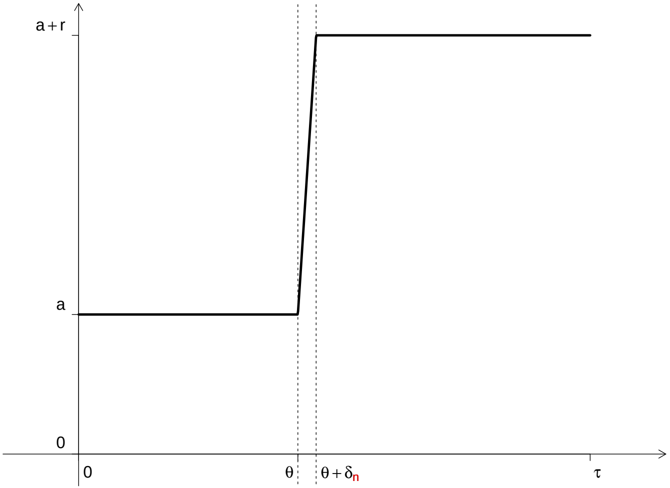

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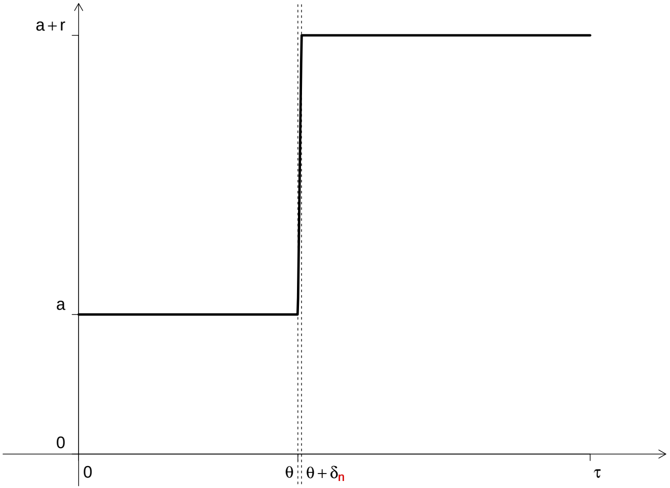

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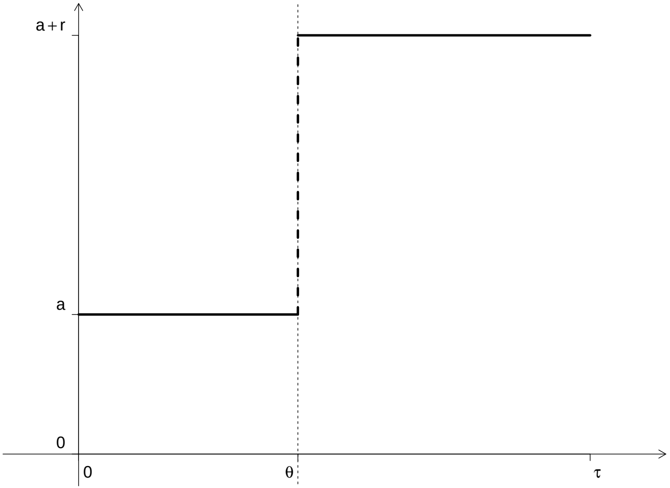
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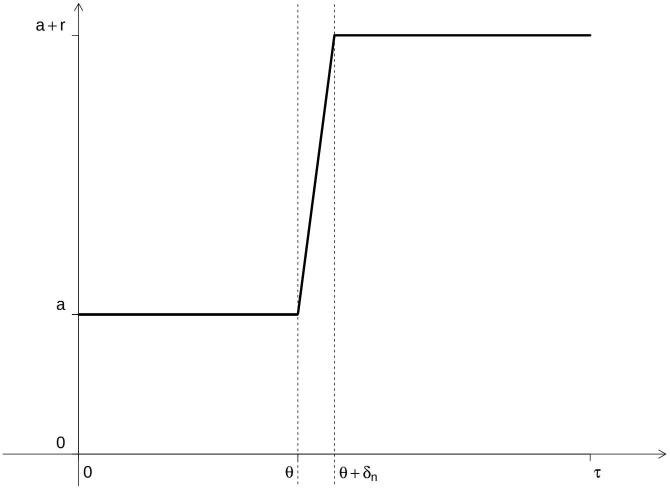

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$$\mathbf{L}\big(\theta, X^{(n)}\big) = \exp\left\{\sum_{j=1}^{n}\int_{0}^{\tau} \ln \lambda_{\theta}^{(n)}(t)\, \mathrm{d}X_j(t) - n\int_{0}^{\tau}\Big(\lambda_{\theta}^{(n)}(t) - 1\Big)\, \mathrm{d}t\right\}.$$

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The Bayes estimator (BE) $\widetilde{\theta}_n$ for quadratic loss and prior density $p(\cdot)$ (supposed continuous and strictly positive) is

$$\widetilde{\theta}_n = \frac{\int_\alpha^\beta \theta\, p(\theta)\, \mathbf{L}\big(\theta, X^{(n)}\big)\, \mathrm{d}\theta}{\int_\alpha^\beta p(\theta)\, \mathbf{L}\big(\theta, X^{(n)}\big)\, \mathrm{d}\theta}.$$

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- the BEs are asymptotically efficient.

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- $\mathbf{E}\,\zeta^2$ is an asymptotic lower bound on MSEs of all estimators.

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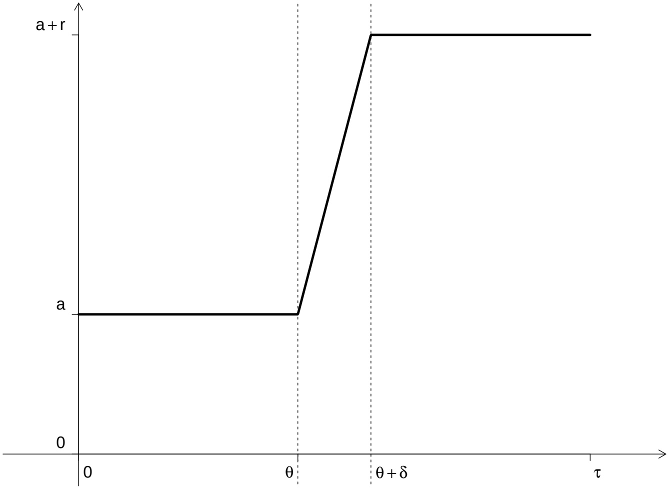
- When the trajectories of these processes are discontinuous, the weak convergence takes place in the space $\mathscr{D}_0(\mathbb{R})$ of càdlàg functions on $\mathbb{R}$ vanishing at $\pm\infty$ equipped with the "usual" Skorokhod topology.



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Theorem

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- They are asymptotically normal $\left(\text{with classic rate } \frac{1}{\sqrt{n}}\right)$:

$$\sqrt{n}\big(\widehat{\theta}_n-\theta\big) \Rightarrow \mathcal{N}\Big(0,\frac{1}{I}\Big) \quad \text{and} \quad \sqrt{n}\big(\widetilde{\theta}_n-\theta\big) \Rightarrow \mathcal{N}\Big(0,\frac{1}{I}\Big).$$

- The convergence of moments in these convergences in law holds.
- Both the MLE and the BEs are asymptotically efficient.

Choosing $\varphi_n = \dfrac{1}{\sqrt{n}}$ and denoting

$$Z_I(u) = \exp\left\{ u\,\Delta - \frac{u^2}{2}\,I \right\}, \quad u \in \mathbb{R},$$

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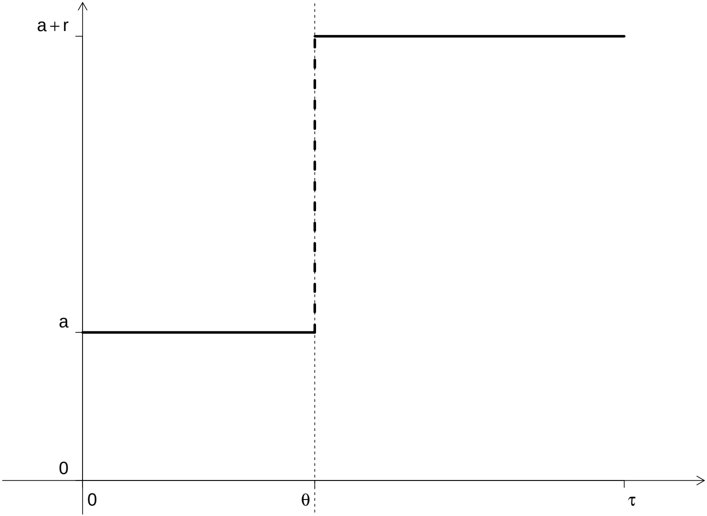

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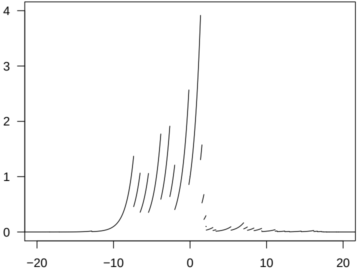

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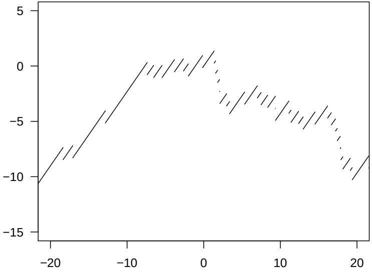

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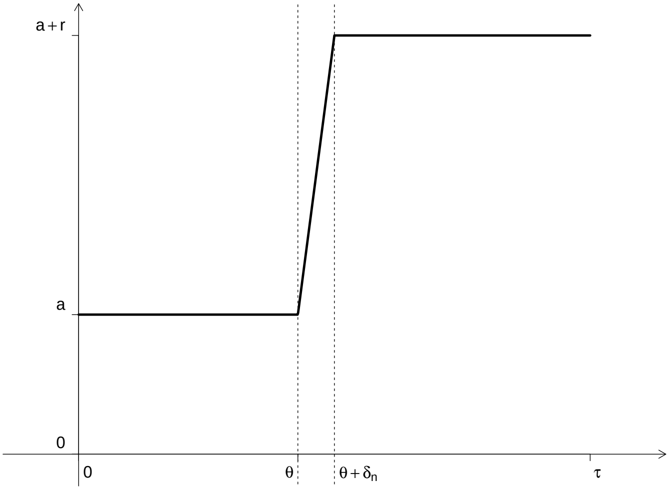
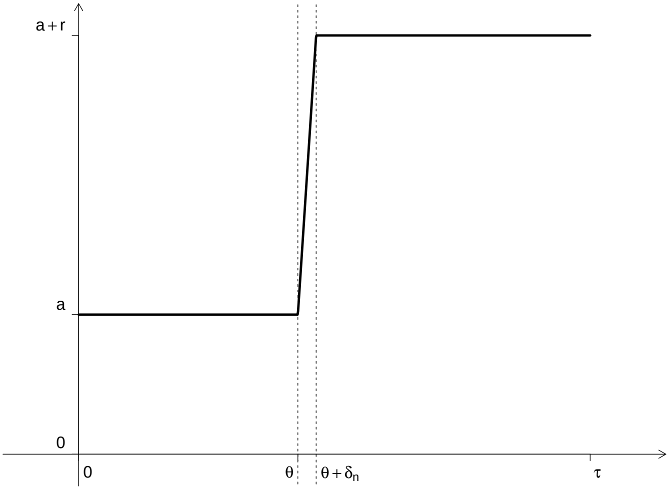
Choosing $\varphi_n = \dfrac{1}{n}$, we have the weak convergence in the space $\mathscr{D}_0(\mathbb{R})$ of the normalized likelihood ratio $Z_n(\cdot)$ to the limit likelihood ratio $Z_{a,a+r}(\cdot)$.



$$\lambda_\theta^{(n)}(t) = a + \frac{r}{\delta_n}\,(t-\theta)\,\mathbb{1}_{[\theta,\theta+\delta_n)}(t) + r\,\mathbb{1}_{[\theta+\delta_n,\tau]}(t), \quad 0 \le t \le \tau.$$



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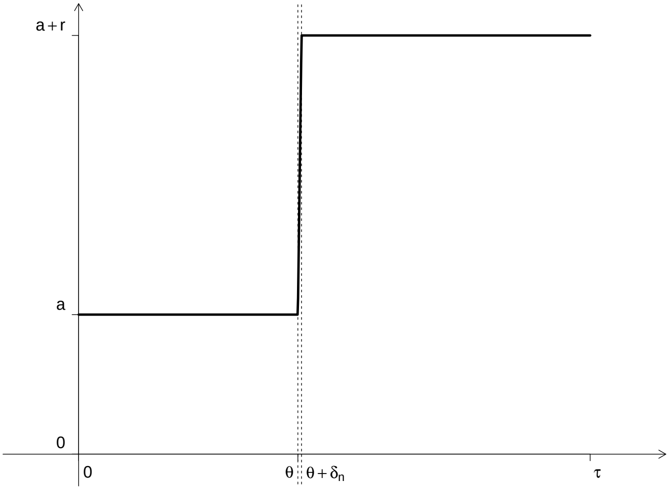

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Naive calculation of the Fisher information for our model yields

$$I_n(\theta) = n \int_0^T \frac{\left(\dot{\lambda}_\theta(t)\right)^2}{\lambda_\theta(t)}\,\mathrm{d}t = \frac{n}{\delta_n}\, r\, \ln\Big(\frac{a+r}{a}\Big) = \frac{n}{\delta_n}\, F.$$

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In fact, the behavior of the estimators is essentially different in the following two cases:

- $n\,\delta_n \to +\infty$ (slow case),
- $n\,\delta_n \to 0$ (fast case).

Theorem

Suppose $n\,\delta_n \to +\infty$.

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Suppose $n\,\delta_n \to +\infty$. Then:

- The MLE and the BEs are consistent.
- They are asymptotically normal with rate $\sqrt{\dfrac{\delta_n}{n}}$:

$$\sqrt{\frac{n}{\delta_n}}\big(\widehat{\theta}_n - \theta\big) \Rightarrow \mathcal{N}\Big(0, \frac{1}{F}\Big) \quad \text{and} \quad \sqrt{\frac{n}{\delta_n}}\big(\widetilde{\theta}_n - \theta\big) \Rightarrow \mathcal{N}\Big(0, \frac{1}{F}\Big).$$

- The convergence of moments in these convergences in law holds.
- Both the MLE and the BEs are asymptotically efficient.

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The model is still regular (LAN): choosing $\varphi_n = \sqrt{\dfrac{\delta_n}{n}}$ and denoting, as before,

$$Z_F(u) = \exp\left\{u\,\Delta - \frac{u^2}{2}\,F\right\}, \quad u \in \mathbb{R},$$

where $\Delta \sim \mathcal{N}(0,F)$, we have the weak convergence in the space $\mathscr{C}_0(\mathbb{R})$ of the normalized likelihood ratio $Z_n(\cdot)$ to the limit likelihood ratio $Z_F(\cdot)$.

Recall:

$$Z_{a,a+r}(u) = \begin{cases} \exp\Big\{\ln\big(\frac{a}{a+r}\big)\, Y_{a+r}(u) + r\,u\Big\}, & \text{if } u \geq 0, \\ \exp\Big\{\ln\big(\frac{a+r}{a}\big)\, Y_{a}(-u) + r\,u\Big\}, & \text{if } u \leq 0, \end{cases}$$

$$\xi_{a,a+r} = \operatorname*{argsup}_{u\in\mathbb{R}} Z_{a,a+r}(u),$$

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However, the trajectories of $Z_n(\cdot)$ are continuous, while those of $Z_{a,a+r}(\cdot)$ are discontinuous, and so $Z_n(\cdot)$ cannot converge to $Z_{a,a+r}(\cdot)$ in the space $\mathscr{D}_0(\mathbb{R})$ equipped with the “usual” Skorokhod topology.

Four topologies introduced by Skorokhod (1956):

$$U \Rightarrow J_1 \begin{array}{c} \nearrow \; J_2 \; \searrow \\ \searrow \; M_1 \; \nearrow \end{array} M_2.$$

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$$d^{(J_1)}(f,g) = \inf_{\nu} \Big[\sup_{u\in\mathbb{R}} \big| f(u) - g\big(\nu(u)\big) \big| + \sup_{u\in\mathbb{R}} \big| u - \nu(u) \big| \Big],$$

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with inf taken over all parametric representations $\big(t_1(\cdot), y_1(\cdot)\big)$ and $\big(t_2(\cdot), y_2(\cdot)\big)$ of the graphs Γ_f and Γ_g of f and g.

Modulus of continuity of the topology J_1:

$$\Delta_h^{(J_1)}(f) = \sup \min\bigl\{|f(u) - f(u')|, |f(u) - f(u'')|\bigr\} + \sup_{|u|>1/h} |f(u)|$$

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Modulus of continuity restricted to an interval $[A, B] \subset \mathbb{R}$:

$$\Delta_h^{(M_1)}(f; [A, B]) = \sup_{u\in[A,B]} \; \sup_{u-h\leq u'\leq u\leq u''\leq u+h} d\big(f(u); [f(u'), f(u'')]\big).$$

Let the processes $Y_n(\cdot)$ and $Y(\cdot)$ with trajectories in $\mathscr{D}_0(\mathbb{R})$. Then $Y_n(\cdot)$ converges weakly to $Y(\cdot)$ in the topology M_1 if and only if:

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Suppose that:

- the finite dimensional distributions of the normalized likelihood ratio $Z_n(\cdot)$ converge to those of some $Z(\cdot)$ having a unique argsup;
- $\mathbf{E}\left|Z_n^{1/2}(u) - Z_n^{1/2}(v)\right|^2 \leq C|u-v|$;
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- $\mathbf{P}\Big(\Delta_h^{(M_1)}\big(Z_n^{1/2};[\ell,\ell+1]\big) > h^{\gamma_1}\Big) \leq B(\ell)\,h^{\gamma_2}$ $(\forall\ 0<h<h_0)$, with $B(\cdot)$ of at most polynomial growth.

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Then $Z_n(\cdot)$ converges weakly to $Z(\cdot)$ in the topology M_1, and the properties of the MLE follow.

For our model, in the fast case, we show that:

$\exists\ \gamma, h_0, C > 0$ such that $\forall\ 0 < h < h_0$ it holds

$$\mathbf{P}\Big(\Delta_h^{(M_1)}\big(Z_n^{1/2};[\ell,\ell+1]\big) > h^{\gamma}\Big) \leq C\, h^{2\gamma}.$$

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This yields the weak convergence of $Z_n(\cdot)$ to $Z_{a,a+r}(\cdot)$ in the topology M_1 and the properties of the MLE.

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- $\Delta_h^{(U)}(f) = \sup_{u,v\in[a,b]\,:\,u\leq v\leq u+h} \bigl|f(v) - f(u)\bigr|$.

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Introduce:

- modulus of increase $\Delta_h^{+}(f) = \sup_{u,v\in[a,b]\,:\,u\leq v\leq u+h} \big(f(v)-f(u)\big)_+\,,$
- modulus of decrease $\Delta_h^{-}(f) = \sup_{u,v\in[a,b]\,:\,u\leq v\leq u+h} \big(f(v)-f(u)\big)_-\,.$

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Corollary: If f is a monotonic, then $\Delta_h^{(M_1)}(f) = 0$!

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Corollary: If f is a monotonic, then $\Delta_h^{(M_1)}(f) = 0$!

Proposition: If the jumps of f have the same sign and are summable, then

$$\Delta_h^{(M_1)}(f) \leq \Delta_h^{(U)}(f_{\mathrm{c.}}),$$

where $f_{\mathrm{c.}}$ is the continuous part of f.

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Proposition:

- f is L-Lipschitz increasing $\iff$ $\Delta_h^+(f) \leq Lh$,
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Proposition: Let f be continuous and piecewise differentiable. Then:

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Generalizes to càdlàg piecewise differentiable functions with the convention that the derivative is infinite at jump points:

$$f'(t) = \begin{cases} +\infty, & \text{if } f(t) - f(t-) > 0, \\ -\infty, & \text{if } f(t) - f(t-) < 0. \end{cases}$$

Thank you for your attention!