Asymptotic analysis in some problems with fractional Brownian motion

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Contribution in a nutshell

Stochastic processes

$\Downarrow$

Integral equations

$\Downarrow$

Asymptotic analysis

This talk: a brief survey of a few such problems with fBm

Fractional Brownian motion

Definition

The fBm $B^H = (B^H_t; t \geq 0)$ is the centered Gaussian process with

$$\operatorname{cov}(B^H_t, B^H_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

where $H \in (0,1)$ is the Hurst parameter.

- H-self similar Gaussian process with stationary increments

$$B^H_t - B^H_s \sim N(0, |t-s|^{2H})$$

- H-Hölder paths
- neither a semimartingale, nor a Markov process for $H \neq \frac{1}{2}$
- long range dependence for $H > \frac{1}{2}$

$$\sum_{n=1}^{\infty} \text{Cov}\left(\Delta B^H_1, \Delta B^H_n\right) = \infty$$

The eigenvalue problem

The eigenvalue problem

- A random process $X = (X_t, t \in [0, T])$ with $\mathsf{E} X_t \equiv 0$ and covariance kernel

$$K(s,t) = \mathsf{E} X_s X_t.$$

- Covariance operator

$$(Kf)(t) := \int_0^T K(s,t) f(s) ds.$$

- Find all pairs (λ, φ) with $\lambda \in \mathbb{R}$ which solve the equation

$$K\varphi = \lambda\varphi.$$

The eigenvalue problem

General theory

Countably many solutions (λ_n, φ_n) if e.g. $K \in L^2([0,T]^2)$

- the eigenvalues λ_n are nonnegative and $\lambda_n \searrow 0$
- the eigenfunctions φ_n form orthogonal basis in $L^2([0,T])$

Applications:

- Karhunen-Loeve approximation
- Statistical inference (nonparametric tests, etc.)
- Filtering and detection problems
- Small L^2-ball probabilities
- Sampling from heavy tailed distributions
- etc.

The eigenvalue problem

Example (Brownian motion)

For $K(s,t) = \min(s,t)$ with $s, t \in [0,1]$

$$\lambda_n = \nu_n^{-2}, \quad \varphi_n(t) = \sqrt{2}\sin(\nu_n t), \quad n = 1, 2, ...$$

where $\nu_n = (n - \frac{1}{2})\pi$.

Some other processes with exact solutions:

- Brownian bridge
- Demeaned Bm
- OU process
- integrated Bm and bridge
- etc.

The approach: reduction to BVP for ODEs

Eigenvalue problem for the fBm

Find all solutions (λ, φ) to the integral equation

$$\int_0^1 K(s,t)\varphi(s)ds = \lambda\varphi(t), \quad t \in [0,1]$$

for $K(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$ with $H \in (0,1)$.

Theorem (Bronski 03)

$$\lambda_n = \frac{\sin(\pi H)\Gamma(2H+1)}{(n\pi)^{2H+1}} + o\left(n^{-\frac{(2H+2)(4H+3)}{4H+5}+\delta}\right) \quad \text{as } n \to \infty.$$

Other proofs: Nazarov and Nikitin 04, Luschgy and Pagès 04.

Eigenvalue problem for the fBm

Theorem (Chigansky and Kleptsyna, 2018)

1. *The eigenvalues satisfy*

$$\lambda_n = \sin(\pi H)\Gamma(2H+1)\nu_n^{-2H-1}, \quad n = 1, 2, \ldots$$

where the sequence ν_n *has the asymptotics*

$$\nu_n = \big(n - \tfrac{1}{2}\big)\pi - \frac{(H - \frac{1}{2})^2}{H + \frac{1}{2}}\frac{\pi}{2} + O(n^{-1}), \quad n \to \infty.$$

- Exact second order asymptotics.

Eigenvalue problem for the fBm

Theorem (continued)

2. *The eigenfunctions satisfy*

$$\varphi_n(t) = \sqrt{2}\sin\left(\nu_n t + \eta_H\right) + \int_0^\infty \left(f_0(u)e^{-t\nu_n u} + f_1(u)e^{-(1-t)\nu_n u}\right)du + n^{-1}r_n(t), \quad t \in [0,1]$$

where f_0 *and* f_1 *are explicit (cumbersome) and*

$$\eta_H = \frac{1}{4}\frac{(H-\frac{1}{2})(H-\frac{3}{2})}{H+\frac{1}{2}} \quad \textit{and} \quad \sup_{t\in[0,1]}\left|r_n(t)\right| \le C(H).$$

- uniform approximation: oscillatory term + boundary layer terms + residual

Eigenvalue problem for the fBm

Other "fractional" processes:

- fractional "noise" (formal derivative of fBm)
- fractional Brownian bridge
- fractional OU process
- integrated fBm
- mixed fBm
- Riemann-Liouville process

An application: small L^2-ball probabilities problem

An application: small ball probabilities

Small ball probabilities problem:

For a given process $X = (X_t, t \in [0,1])$ and a norm $\|\cdot\|$ find the asymptotics of $\mathsf{P}(\|X\| \le \varepsilon)$ as $\varepsilon \to 0$.

- The most studied is the Gaussian case with L^2-norm:

$$\mathsf{P}(\|X\|_2 \le \varepsilon) = \mathsf{P}\Big(\sum_{n=1}^{\infty} \lambda_n Z_n^2 \le \varepsilon^2\Big) \text{ with } Z_n \overset{\text{i.i.d}}{\sim} N(0,1)$$

Example (Bm, Cameron-Martin 44)

$$\mathsf{P}(\|B\|_2 \le \varepsilon) = \frac{4}{\sqrt{\pi}} \varepsilon \exp\big(-\frac{1}{8}\varepsilon^{-2}\big)(1 + o(1)), \quad \varepsilon \to 0.$$

- Some milestones: Sytaja 75 (Laplace transform), Li 92 (comparison theorem), Dunker, Lifshits and Linde 98 (checkable conditions), ...

An application: small ball probabilities

Nikitin and Nazarov 2004:

- The asymptotics of $-\log \mathsf{P}(\|X\|_2 \leq \varepsilon)$ is determined by the first order asymptotics of λ_n

Corollary (Bronski 03)

For any $H \in (0, 1)$,

$$-\log \mathsf{P}(\|B^H\|_2 \leq \varepsilon) = \beta(H)\varepsilon^{-1/H}(1 + o(\varepsilon)), \quad \varepsilon \to 0,$$

where

$$\beta(H) = H \frac{\big(\sin(\pi H)\Gamma(2H+1)\big)^{1/(2H)}}{\big((2H+1)\sin\frac{\pi}{2H+1}\big)^{1+1/(2H)}}$$

An application: small ball probabilities

Nikitin and Nazarov 2004:

- The asymptotics of $\mathsf{P}(\|X\|_2 \le \varepsilon)$ is determined by the second order asymptotics of λ_n, up to a mult. "distortion" constant.

Corollary (Chigansky and Kleptsyna 2018)

For all $H \in (0, 1)$,

$$\mathsf{P}(\|B^H\|_2 \le \varepsilon) = C_d(H)\varepsilon^{\gamma(H)} \exp\left(-\beta(H)\varepsilon^{-1/H}\right)(1 + o(1)),\ \varepsilon \to 0$$

where $C_d(H)$ *is (yet unknown) distortion constant and*

$$\gamma(H) = \frac{1}{2H}\left(3/4 + H^2 + \frac{1}{2}\frac{1-2H}{(1+2H)^2}\right).$$

Fredholm integral equations

Fredholm integral equations

First kind:

$$\int_0^T K(s,t)g(s)ds = f(t), \quad t \in [0,T]$$

Second kind:

$$\varepsilon g(t) + \int_0^T K(s,t)g(s)ds = f(t), \quad t \in [0,T]$$

- $\varepsilon \in \mathbb{R}_+$ and $T \in \mathbb{R}_+$ are parameters
- $f(t)$ is a given "forcing" function
- $K(s,t)$ is a kernel (e.g. covariance function)

Fredholm integral equations

General theory

Existence and uniqueness of solutions in appropriate spaces depending on K and f. The solutions are rarely explicit.

A frequent objective is to find the asymptotics of a specific functional of the solution w.r.t. a relevant parameter:

- Large time $T \to \infty$ (finite section limits)
- Small parameter $\varepsilon \to 0$ (singular perturbation)

An application: filtering problem

An application: filtering problem

- State process $X = (X_t; t \in [0, T])$
- Observation process (additive Gaussian noise)

$$Y_t = \mu \int_0^t X_t dt + \sqrt{\varepsilon} V_t, \quad t \in [0, T]$$

where V is a process independent of X, μ and ε are constants

- The optimal estimator of X_t and its MSE

$$\widehat{X}_t = \mathsf{E}(X_t | \mathcal{F}_t^Y) \quad \text{and} \quad P_t = \mathsf{E}(X_t - \widehat{X}_t)^2.$$

An application: filtering problem

- The optimal filter is

$$\widehat{X}_T = \frac{1}{\sqrt{\varepsilon}} \int_0^T g(s) dY_s$$

- The weight function $g(s)$ solves integro-differential equation

$$\frac{\partial}{\partial s} \int_0^T g(r) \frac{\partial}{\partial r} K_V(r,s) dr + \frac{\mu^2}{\varepsilon} \int_0^T K_X(r,s) g(r) dr = \frac{\mu}{\sqrt{\varepsilon}} K_X(s,T), \quad s \in [0,T],$$

with the covariance kernels

$$K_X(s,t) = \mathrm{Cov}(X_s, X_t) \quad \text{and} \quad K_V(s,t) = \mathrm{Cov}(V_s, V_t).$$

- The minimal MSE is given by the functional

$$P_T = \frac{\sqrt{\varepsilon}}{\mu} \left(\frac{\partial}{\partial s} \int_0^T g(r) \frac{\partial}{\partial r} K_V(r,s) dr \right)_{\big| s := T}.$$

The Markov case: Kalman & Bucy

- Observation disturbance V is the Brownian motion
  $\Longrightarrow$ integral equation of second kind
- Markov state process (OU)

$$X_t = \beta \int_0^t X_s ds + W_t, \quad t \in [0, T]$$

  where W is a Brownian motion.
  $\Longrightarrow$ reduction of the integral equation to the Riccati ODE

$$\dot{P}_t = 2\beta P_t + 1 - (\mu/\sqrt{\varepsilon})^2 P_t^2$$

The Markov case: Kalman & Bucy

Elementary analysis of the Riccati ODE

$$\boxed{\dot{P}_t = 2\beta P_t + 1 - (\mu/\sqrt{\varepsilon})^2 P_t^2}$$

reveals that

- the steady state error exists and is given by

$$P_T\Big(\beta, \frac{\mu}{\sqrt{\varepsilon}}\Big) \xrightarrow[T\to\infty]{} \frac{\beta + \sqrt{\beta^2 + \mu^2/\varepsilon}}{\mu^2/\varepsilon}$$

- the small noise asymptotics is

$$P_T\Big(\beta, \frac{\mu}{\sqrt{\varepsilon}}\Big) = \frac{\sqrt{\varepsilon}}{\mu}\big(1 + o(1)\big) \quad \text{as } \varepsilon \to 0, \quad \forall T > 0.$$

[Q]: Kalman-Bucy model with fBm's W^{H_1} and V^{H_2} ...?

Kalman-Bucy model with fractional noises

Theorem (Afterman, Ch., Kleptsyna, Marushkevych 22)

For the Kalman-Bucy model with fBm's W^{H_1} and V^{H_2}, the steady state error exists:

$$P_\infty\Big(\beta, \frac{\mu}{\sqrt{\varepsilon}}\Big) = \lim_{T\to\infty} P_T\Big(\beta, \frac{\mu}{\sqrt{\varepsilon}}\Big),$$

and, for any $T > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-\nu} P_T\Big(\beta, \frac{\mu}{\sqrt{\varepsilon}}\Big) = P_\infty(0, \mu) \quad \textit{with } \nu = \frac{H_1}{1 + H_1 - H_2}.$$

- Asymptotic error increases with roughness of the noises

Kalman-Bucy model with fractional noises

A more detailed answer in some meaningful special cases such as

Theorem (fractional state + white observation noises)

Let $H := H_1 \in (0,1)$ *and* $H_2 = \frac{1}{2}$, *then*

$$P_\infty\Big(\beta, \frac{\mu}{\sqrt{\varepsilon}}\Big) = \frac{\varepsilon}{\mu^2}\left(\frac{1}{\pi}\int_0^\infty \theta(t; H, \tfrac{1}{2})dt + \beta + 2\mathrm{Re}(z_0)\mathbf{1}_{\{H>\frac{1}{2}\}}\right)$$

Consequently, for any $T > 0$,

$$P_T\Big(\beta, \frac{\mu}{\sqrt{\varepsilon}}\Big) \asymp \frac{\big(\Gamma(2H+1)\sin(\pi H)\big)^{\frac{1}{2H+1}}}{\sin\frac{\pi}{2H+1}}(\varepsilon/\mu^2)^{\frac{2H}{2H+1}}, \quad \text{as } \varepsilon \to 0.$$

- Reduces to the classical spectral formula in the stable case $\beta < 0$, but is valid also in the unstable case $\beta \geq 0$!

Application: statistical inference

Estimation of the Hurst parameter from noisy data

Problem

Estimate parameters $H \in (\frac{3}{4}, 1)$ and $\sigma \in \mathbb{R}_+$ given the sample

$$X_t = \sigma B_t^H + \sqrt{\varepsilon} B_t, \quad t \in [0, T]$$

where fBm B^H and Bm B are independent and $\varepsilon > 0$ is known.

Theorem (Shepp 66, Cheridito 01)

The measures induced by X and $\sqrt{\varepsilon} B$ are equivalent iff $H > \frac{3}{4}$.

Estimation of the Hurst parameter from noisy data

Theorem (Chigansky and Kleptsyna, 2023)

The parameters $H \in (\frac{3}{4}, 1)$ *and* $\sigma \in (0, \infty)$ *can be estimated at the optimal local minimax rates*

$$\varepsilon^{1/(4H-2)} \quad \textit{and} \quad \varepsilon^{1/(4H-2)} \log \varepsilon^{-1}.$$

respectively, as $\varepsilon \to 0$.

Proof: verification of the "singular" LAN property (as in Brouste and Fukasawa, 18) which hinges on asymptotic analysis of the equation

$$\varepsilon g(t) + \int_0^T K(t-s) g(s) ds = K(t), \quad 0 < s < t < T$$

with the weakly singular kernel (of fractional "noise")

$$K(t) = \sigma H(2H-1)|t|^{2H-2}.$$

Bon anniversaire, LMM!