

# HJB equation on process space

## Application to mean field control

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**HJB equation in  $\mathbb{R}^d$  :**

$$\partial_t u + \inf_{a \in A} \left\{ b(\cdot, a) \cdot Du + \frac{1}{2} \sigma \sigma^\top(\cdot, a) : D^2 u - k(\cdot, a) u + f(\cdot, a) \right\} = 0$$
$$u|_{t=T} = g$$

**Control process**  $\alpha$  prog. meas. with values in  $A$

**Controlled state process** driven by BM  $W$  in  $\mathbb{R}^d$  :

$$X_t^{t,x} = x \text{ and } dX_s^{t,x} = b(s, X_s^{t,x}, \alpha_s) ds + \sigma(s, X_s^{t,x}, \alpha_s) dW_s$$

**Stochastic control problem**

$$V(t, x) = \inf_{\alpha} \mathbb{E} \left[ g(X_T^{t,x}) \beta_{t,T} + \int_t^T \beta_{t,s} f(s, X_s^{t,x}, \alpha_s) ds \right], \beta_{t,s} := e^{-\int_t^s k(r, X_r^{t,x}, \alpha_r) dr}$$

Under standard assumptions,  $V$  **viscosity solution** of the HJB equation

# Viscosity solutions (Crandal & Lions)

Consider second order parabolic PDE

$$F(t, x, u(t, x), Du(t, x), D^2u(t, x)) = 0$$

$$\underline{A}u(t, x) := \left\{ \varphi \in C^2 : 0 = (u^* - \varphi)(t, x) = \max(u^* - \varphi) \right\}$$

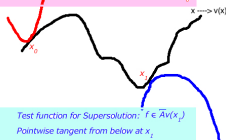
**Def.**  $u$  viscosity subsol if  $F[\phi] \leq 0$  for all  $\phi \in \underline{A}u(t, x)$

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**Def.**  $u$  viscosity supersol if  $F[\phi] \geq 0$  for all  $\phi \in \overline{A}u(t, x)$

$f \in \underline{A}v(x_0)$  Test function for Subsolution:

Pointwise tangent from above at  $x_0$



Test function for Supersolution:  $f \in \overline{A}v(x_1)$

Pointwise tangent from below at  $x_1$

Requires

LOCAL COMPACTNESS OF UNDERLYING SPACE

**Consistency** : For  $u \in C^{1,2}$ ,  $u$  classical sol. **iff**  $u$  viscosity sol.

**Stability** :  $v_\varepsilon$  be visco supersol, loc bdd in  $(\varepsilon, t, x)$ . Then

$$\underline{v}(t, x) := \liminf_{(\varepsilon, t', x') \rightarrow (0, t, x)} v_\varepsilon(t', x')$$
 is a visco supersol

**Uniqueness** ...

# Comparison of smooth viscosity sub- and supersolutions

General class including 2nd order equations :

$$F(x, u, Du, D^2u) = 0, \quad \text{on open } \mathcal{O} \subset \mathbb{R}^d, \quad F \text{ str. } \nearrow \text{ in } u, \searrow \text{ in } D^2u$$

- Assume  $\mathcal{O}$  bdd for simplicity
- If  $U - V \leq 0$  on  $\partial\mathcal{O}$  with  $M := \max(U - V) > 0$ . Then  
 $M = (U - V)(x_0)$  for some  $x_0 \in \mathcal{O}$  (by loc. compactness)
- If  $U, V$  classical subsol and supersol, then the FOC and SOC imply

$$\begin{aligned} F(x_0, U(x_0), DU(x_0), D^2U(x_0)) &\leq 0 \leq F(x_0, V(x_0), DV(x_0), D^2V(x_0)) \\ &= F(x_0, U(x_0) - M, DV(x_0), D^2V(x_0)) \\ &< F(x_0, U(x_0), DU(x_0), D^2U(x_0)) \end{aligned}$$

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# Comparison of viscosity sub- and supersolutions : 1st order

If  $U \in \text{USC}$  and  $V \in \text{LSC}$ , use doubling variables :

$$M_n := \max_{x \in \text{cl}(\mathcal{O}), y \in \text{cl}(\mathcal{O})} U(x) - V(y) - n|x - y|^2, \text{ attained at } (x_n, y_n)$$



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$\varphi$  test function for  $U$  at  $x_n$

$$F(x_n, U(x_n), \underbrace{D\varphi(x_n)}_{=n(x_n - y_n)}) \leq 0$$

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• For first order  $F(x, u, Du) = 0$  :

$$F(x_n, U(x_n), n(x_n - y_n)) \leq F(y_n, U(x_n) - M_n - n|x_n - y_n|^2, n(x_n - y_n)) \dots$$

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- Because  $D^2\varphi = nI_d$  and  $D^2\psi = -nI_d$ , we need in addition the Crandall-Ishii's lemma...

Many extensions starting from the HJB equation on Hilbert spaces

- Extension by Lions, Swiech, Gozzi, ...
- Crandall-Lions and Li-Yong use

Test functions of the form  $\varphi + \phi$  with **nonsmooth**  $\phi$

Consider the non-Markov version of the previous control problem

$$X_{\wedge t}^{t,\omega} = \omega_{\wedge t} \text{ and } dX_s^{t,\omega} = b_s(X_{\wedge s}^{t,\omega}, \alpha_s) ds + \sigma_s(X_{\wedge s}^{t,\omega}, \alpha_s) dW_s$$

## Stochastic control problem

$$V_t(\omega) := \sup_{\alpha} \mathbb{E} \left[ g(X_{\wedge T}^{t,\omega}) \beta_{t,T} + \int_t^T \beta_{t,s} f(s, X_{\wedge s}^{t,\omega}, \alpha_s) ds \right], \quad \beta_{t,s} := e^{-\int_t^s k_r^\alpha dr}$$

## HJB characterization (Under standard assumptions)

$V$  is the unique viscosity solution of the path-dependent HJB equation

$$\partial_t V + \inf_{a \in A} \left\{ b(\cdot, a) \cdot \partial_\omega V + \frac{1}{2} \sigma \sigma^\top(\cdot, a) : \partial_{\omega\omega}^2 V - k(\cdot, a) V + f(\cdot, a) \right\} = 0$$

- BSDE (pardoux-Peng), GBSDE (Peng) and 2BSDE (Soner-NT-Zhang)
- Ekren-NT-Zhang : Test functions  $\mathbb{E}$ -tangent
- Jianjun Zhou : back to standard def (pointwise tangency) using Ekeland-Borwein-Preis variational Lemma

# Stochastic control of population of symmetric particles

**Finite population**  $X^1, \dots, X^N$  driven by independent BMs  $W^i$  in  $\mathbb{R}^d$

$$X_t^i = x^i \text{ and } dX_s^i = b(s, X_s^i, \underbrace{m_N(X_s)}_{\text{empirical measure}}, \alpha_s^i) ds + \sigma(\dots) dW_s^i$$

**Stochastic control problem**  $:= \frac{1}{N} \sum_{i=1}^N \delta_{X_s^i}$

$$V_N(t, x) = \sup_{\alpha^1, \dots, \alpha^N} \mathbb{E} \left[ \sum_{i=1}^N g(X_T^i, m_N(X_T)) + \int_t^T f(X_s^i, m_N(X_s), \alpha_s^i) ds \right]$$



$$\text{HJB} : 0 = \partial_t V_N + \sum_{i=1}^N \inf_{a \in A} \left\{ b(x^i, m_N(x), a) \cdot D_{x^i} V_N + \frac{1}{2} \sigma \sigma^T(\dots) : D_{x^i x^i}^2 V_N + f(\dots) \right\}$$

Then  $V_N(t, x) \rightarrow V(t, m) := \sup_{\alpha} \mathbb{E} \left[ g(X_T, \mathcal{L}_{X_T}) + \int_t^T f(s, X_s, \alpha_s) ds \right]$   
 where  $\mathcal{L}_{X_t} = m$  and  $dX_s = b(s, X_s, \mathcal{L}_{X_s}, \alpha_s) ds + \sigma(\dots) dW_s$

and  $V$  solution of the HJB equation on the Wasserstein space

$$0 = \partial_t V + \int \inf_{a \in A} \left\{ b(x, m, a) \cdot \partial_L V + \frac{1}{2} \sigma \sigma^T(\dots) : \partial_x \partial_L V + f(\dots) \right\} m(dx)$$



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# Mean field control with common noise

Suppose particles dynamics affected by a **common BM**  $W^0$  :

$$X_t^i = x^i \text{ and } dX_s^i = b(s, X_s^i, m_N(X_s), \alpha_s^i) ds + \sigma(\dots) dW_s^i + \sigma^0(\dots) dW_s^0$$

Then, the corresponding mean field limit is :

$$\mathcal{L}(X_t) = m \text{ and } dX_s = b(s, X_s, \mathcal{L}_{X_s|W_0}, \alpha_s) ds + \sigma(\dots) dW_s + \sigma^0(\dots) dW_s^0$$

Then, mean field control problem with common noise

$$V(t, m) := \sup_{\alpha} \mathbb{E} \left[ g(X_T, \mathcal{L}_{X_s|W_0}) + \int_t^T f(s, X_s, \alpha_s) ds \right]$$

and  $V$  solution of a **second order HJB eq. on the Wasserstein space ...**

- 1st order equation on the Wasserstein space :
  - involving  $\partial_L u$  only : Bertucci, Cardaliaguet-Quincampoix, Conforti, Kraaij-Tonon, Feng-Katsoulakis, Gangbo, Nguyen-Tudorascu, Gangbo-Tudorascu, Jimenez, Marigonda-Quincampoix
  - involving  $\partial_L u$  and  $\partial_x \partial_L u$  : Wu-Zhang, Cosso-Gozzi-Kharroubi-Pham-Rosestolato, Talbi-NT-Zhang, Burzoni-Ignazio-Reppen-Soner, Soner-Yan
- 2nd order equation on the Wasserstein space :
  - Bayraktar-Ekren-Zhang extend the Crandal Ishii's lemma to the present context
  - Gangbo-Mayorga-Swiech, Mayorga-Swiech, Daudin-Seeger : lifting on the Hilbert space of random variables

## Our main objective is

### Underlying space is the set of random processes

- Natural framework, [Next Example 1]
- Test functions will have a smooth component and a singular one

$$\varphi + \phi$$

Comparison result by (several) **doubling variables argument only**, thus avoiding the Crandall-Ishii lemma [see Next Example 2]

# Example 1 : Mean field control with common noise

Let  $W^0$  BM and consider the MF control pb.

$$\mathfrak{V}(0, \mu) := \inf_{\alpha} \mathbb{E} [G(\mathcal{L}_{X_T | \mathcal{F}_T^{W^0}})], \text{ where } G : \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$$

$$\text{and } \mathcal{L}_{X_0 | \mathcal{F}_T^{W^0}} = \mu, \quad dX_t = B(X_t, \mathcal{L}_{X_t | \mathcal{F}_T^{W^0}}, \alpha_t) dt + \sigma^0 dW^0, \quad t \geq 0$$

Define  $b : \Omega \times \mathbb{R}^d \times \mathbb{L}^2(\mathbb{R}^d) \times A \longrightarrow \mathbb{R}^d$  and  $g : \Omega \times \mathbb{L}^2(\mathbb{R}^d) \longrightarrow \mathbb{R}$  :

$$b(\omega, x, \underline{\xi}, a) := B(x, \mathcal{L}_{\xi | \mathcal{F}_t^{W^0}}(\omega), a) \text{ and } g(\omega, \underline{\xi}) := G(\mathcal{L}_{\xi | \mathcal{F}_t^{W^0}}(\omega))$$

where we denote  $\underline{\xi}$  to emphasize dependence on the r.v.  $\xi$ .

$\implies$  Control problem on the space of r.v.

$$\mathfrak{V}(0, \mu) = V(0, X_0) := \inf_{\alpha} \mathbb{E} [g(X_T)]$$

$$\text{where } dX_t = b(X_t, \underline{X}_t, \alpha_t) dt + \sigma^0 dW^0, \quad t \geq 0$$

## Example 2 : constant diffusion setting

Consider the mean field control problem

$$\mathfrak{V}(0, \mu) = \inf_{\alpha} G(\mathcal{L}_{X_T})$$

where  $\mathcal{L}_{X_0} = \mu$  and  $dX_t = B(X_t, \alpha_t)dt + dW_t$ ,  $t \geq 0$

Introduce the change of variable

$$x_t := X_t - W_t, \quad b_t(\omega, x, a) := B(x + W_t(\omega), a), \quad g(\underline{x}_T) := G(\mathcal{L}_{x_T + W_T})$$

Here again  $g : \mathbb{L}^2(\mathbb{R}^d) \rightarrow \mathbb{R}$ . Then  $\mathfrak{V}_t(\underline{\xi}) = V_t(\underline{\xi} - \underline{W}_t)$ , where

$V_t$  is a 1st order control problem... on the space of random variables

$$V_t(\underline{\xi}) := \inf_{\alpha} g(\underline{x}_T), \quad \mathcal{L}_{x_0} = \mu \text{ and } dx_t = b_t(x_t, \alpha_t)dt$$

# Doubling variables for the reduced 1st order problem

In the setting of the first order control problem, it is natural to adapt the doubling variables argument with test functions

$$\varphi(t, \underline{\xi}) := V(s_n, \zeta_n) + n \left( |t - s_n|^2 + \mathbb{E}[|\xi - \zeta_n|^2] \right)$$

Recalling our change of variable, this induces the following test function for the initial problem :

$$\phi(t, \underline{\xi}) := V(s_n, \zeta_n) + n \left( |t - s_n|^2 + \mathbb{E}[|\xi - W_t - \zeta_n|^2] \right)$$

$\implies$  introduces a dependence on joint law of  $(\xi, \zeta_n, W)$

$\implies \varphi$  is smooth, BUT  $\phi$  is not  $C^{1,2}$ ... (see later in which sense);  
However, it is absolutely continuous...

The above  $\phi$  is our typical singular component of test function



# The control problem on the space of random processes

We shall consider the control problem on the space of processes

$$V_t(\underline{\xi}) := \inf_{\alpha \in \mathcal{A}_{[t, T]}} g(\underline{X}^{t, \xi, \alpha}) + \int_t^T f_s(\underline{X}^{t, \xi, \alpha}, \underline{\alpha}_s) ds$$

where the controlled state is defined by

$$X_{\wedge t}^{t, \xi, \alpha} = \xi, \quad \text{and for } s \geq t :$$

$$dX_s^{t, \xi, \alpha} = b_s(X^{t, \xi, \alpha}, \alpha_s, \underline{X}^{t, \xi, \alpha}, \underline{\alpha}_s) ds + \sigma_s(X^{t, \xi, \alpha}, \alpha_s, \underline{X}^{t, \xi, \alpha}, \underline{\alpha}_s) dW_s$$

All above functions are maps :

$$[0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{S}^2 \times \mathbb{H}^2 \longrightarrow \mathbb{R}^d, \mathbb{R}d \times d, \text{ or } \mathbb{R}$$

$\mathbb{S}^p$  : continuous  $\mathbb{F}$ -p.m. proc. with  $\|X\|_{\mathbb{S}^p} = \| |X|_{\infty} \|_{L^p(\mathbb{R})} < \infty$

$\mathbb{H}^2$  :  $\mathbb{F}$ -p.m. proc. with  $\|\alpha\|_{\mathbb{H}^2} = \|\alpha\|_{L^2([0, T] \times A)} < \infty$

## Proposition

Under standard Lipschitz conditions,  $V$  is uniformly Lipschitz in  $\xi$ , and locally  $\frac{1}{2}$ -Hölder continuous in  $t$  :

$$|V_t(\underline{\xi}) - V_{t'}(\underline{\xi}')| \leq C \|\xi_{\wedge t} - \xi'_{\wedge t}\|_{\mathbb{S}^2} + C(1 + \|\xi_{\wedge t}\|_{\mathbb{S}^2}) |t - t'|^{\frac{1}{2}}$$

# HJB equation on the process space

Motivated by “horizontal-vertical derivatives” [Dupire '09, Cont & Fournié '13],

## Definition (Smooth maps on the process space)

A map  $\varphi : [t, T] \times \mathbb{S}^p \rightarrow \mathbb{R}$  is  $C^{1,2}$  if  $\varphi \in C^0$  and there exist  $C^0$  maps

$$\begin{aligned}\partial_t \varphi : [t, T] \times \mathbb{S}^p &\rightarrow \mathbb{R} & \text{and} & & \partial_X \varphi : [t, T] \times \mathbb{S}^p &\rightarrow \mathbb{L}^{\frac{p}{p-1}}(\mathbb{R}^d) \\ \partial_{XX} \varphi : [t, T] \times \mathbb{S}^p &\rightarrow \mathbb{L}^{\frac{p}{p-1}}(\mathbb{R}^{d \times d})\end{aligned}$$

such that  $\partial_X \varphi_t(\underline{\xi}), \partial_{XX} \varphi_t(\underline{\xi})$   $\mathcal{F}_t$ -meas. and for all Itô process  $X$

$$d\varphi_t(\underline{X}) = \partial_t \varphi_t(\underline{X})dt + \mathbb{E} \left[ \partial_X \varphi_t(\underline{X}) \cdot dX_t + \frac{1}{2} \partial_{XX} \varphi_t(\underline{X}) : d\langle X \rangle_t \right]$$

Combining with the dynamic programming, we obtain the DPE :

## HJB equation on the process space

$$\partial_t U_t(\underline{\xi}) + \inf_{\alpha} \mathbb{E} \left[ b_t^\alpha \partial_X U_t(\underline{\xi}) + \frac{1}{2} \sigma^\alpha \sigma_t^{\alpha \top} : \partial_{XX} U_t(\underline{\xi}) \right] + f_t(\underline{\xi}, \underline{\alpha}) = 0, t < T, \underline{\xi} \in \mathbb{S}^p$$

## Theorem (Characterization of value function)

Under our assumptions, the value function  $V$  is the unique viscosity solution of the HJB equation with terminal condition  $V_T = g$  in the class of functions satisfying

$$|V_t(\underline{\xi}) - V_{t'}(\underline{\xi}')| \leq C \|\xi_{\wedge t} - \xi'_{\wedge t}\|_{\mathbb{S}^2} + C(1 + \|\xi_{\wedge t}\|_{\mathbb{S}^2}) |t - t'|^{\frac{1}{2}}$$

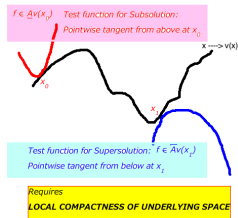
## Theorem (Comparison result)

Let  $U^0, U^1 \in C^0(\mathbb{Q}_0^2)$  be viscosity subsolution and supersolution, respectively, of the HJB equation satisfying

$$|U_t^i(\underline{\xi}) - U_{t'}^i(\underline{\xi}')| \leq C \|\xi_{\wedge t} - \xi'_{\wedge t}\|_{\mathbb{S}^2} + C(1 + \|\xi_{\wedge t}\|_{\mathbb{S}^2}) |t - t'|^{\frac{1}{2}}$$

Then, under our assumptions,  $U_T^0 \leq U_T^1$  on  $\mathbb{Q}_0^2$  implies  $U^0 \leq U^1$  on  $\mathbb{S}^2$

# Test functions on $\mathbb{Q}_0^2$



For  $U \in C^0(\mathbb{Q}_0^2)$  and  $(t, \xi)$

$$\mathfrak{F}^+ U_t(\underline{\xi}) := \left\{ (\varphi, \phi) \in C^{1,2}(\mathbb{Q}_t^6) \times C^+(\mathbb{Q}_t^6) : \right. \\ \left. [U - (\varphi + \phi)]_t(\underline{\xi}) = \sup_{\mathbb{Q}_t^6} [U - (\varphi + \phi)] \right\}$$

$$\mathfrak{F}^- U_t(\underline{\xi}) := \left\{ (\varphi, \phi) \in C^{1,2}(\mathbb{Q}_t^6) \times C^-(\mathbb{Q}_t^6) : \right. \\ \left. [U - (\varphi + \phi)]_t(\underline{\xi}) = \inf_{\mathbb{Q}_t^6} [U - (\varphi + \phi)] \right\}$$

# Singular component of test function

Given  $(s, \zeta) \in \mathbb{Q}_0^p$  and continuous  $(\beta, \gamma) : A \times \mathcal{A} \rightarrow \mathbb{R}^d, \mathbb{R}^{d \times d}$ , denote :

$$\mathcal{I}_t^\alpha(\xi) := \mathcal{I}_t^{\beta, \gamma, s, \zeta, \alpha}(\xi) := \xi_t - \zeta_s - \int_s^t \beta_r^\alpha dr - \int_s^t \gamma_r^\alpha dW_r$$

where  $\beta_r^\alpha := \beta(\alpha_r, \underline{\alpha}_r)$

## Definition

For  $s \in [0, T]$ , we denote  $C^+(\mathbb{Q}_s^p)$  the set of maps of the form :

$$\phi_t(\underline{\xi}) := \inf_{\alpha \in \mathcal{A}_{[s, T]}} \left\{ k \mathbb{E} \left[ |\mathcal{I}_t^\alpha(\xi)|^p + |\mathcal{I}_{t'}^\alpha(\xi')|^p \right] + \int_{t'}^t \psi_r^\alpha dr \right\} \text{ for all } (t, \xi) \in \mathbb{Q}_s^p$$

for some  $k \geq 0$ ,  $\zeta \in \mathbb{S}^p$ ,  $(t', \xi') \in \mathbb{Q}_s^p$ ,  $\beta, \gamma$  as above,  $\psi \in C^0(\mathcal{A}, \mathbb{R})$ .

Moreover, let  $C^-(\mathbb{Q}_s^p) := -C^+(\mathbb{Q}_s^p)$

**Fact :** Any  $\phi \in C^+(\mathbb{Q}_s^p)$  is a.c. wrt Lebesgue

# Viscosity solutions of the HJB equation on $\mathbb{Q}_0^2$

Frozen state process defined for all  $(t, \xi)$  and  $\alpha$

$$\bar{X}_s^{t, \xi, \alpha} := \xi_t + \mathbb{1}_{\{s \geq t\}} \int_t^s b_r^{t, \xi, \alpha} dr + \int_t^s \sigma_r^{t, \xi, \alpha} dW_r, \quad s \geq 0,$$

where  $\psi_s^{t, \xi, \alpha} := \psi(t, \xi_{\wedge t}, \alpha_s, \underline{\xi}_{\wedge t}, \underline{\alpha}_s)$  for  $\psi = b, \sigma$

## Definition

(i)  $U \in C^0(\mathbb{Q}_0^2)$  is a viscosity subsolution of HJB equation if

$$\partial_t \varphi_t(\underline{\xi}) + \liminf_{\delta \rightarrow 0} \inf_{\alpha} \frac{1}{\delta} \int_t^{t+\delta} [H_s(\underline{\xi}_{\wedge t}, \partial_X \varphi_t(\underline{\xi}), \partial_{XX} \varphi_t(\underline{\xi}), \underline{\alpha}_s) + \dot{\phi}_s(\bar{X}^{t, \xi, \alpha})] ds \geq 0$$

for all  $(t, \xi) \in \mathbb{Q}_0^6$  and  $(\varphi, \phi) \in \mathfrak{F}^+ U_t(\xi)$

(ii)  $U \in C^0(\mathbb{Q}_0^2)$  is a viscosity supersolution of HJB equation if

$$\partial_t \varphi_t(\underline{\xi}) + \limsup_{\delta \rightarrow 0} \inf_{\alpha} \frac{1}{\delta} \int_t^{t+\delta} [H_s(\underline{\xi}_{\wedge t}, \partial_X \varphi_t(\underline{\xi}), \partial_{XX} \varphi_t(\underline{\xi}), \underline{\alpha}_s) + \dot{\phi}_s(\bar{X}^{t, \xi, \alpha})] ds \leq 0$$

for all  $(t, \xi) \in \mathbb{Q}_0^6$  and  $(\varphi, \phi) \in \mathfrak{F}^- U_t(\xi)$

(iii)  $U \in C^0(\mathbb{Q}_0^2)$  is a viscosity solution of HJB if ...