Level sets Methods in Pathwise Optimization

Examples of Optimal Stopping and Dynamics utilities

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30 th or 50 th Annviversaire?

Normal pour un probabiliste d'avoir des problèmes avec les changements de temps

Emotion sincère pour un 50 ème anninversaire

- En souvenir de mes années de Professeur de Probabilité de 1973-1979
- De mon premier doctorant, Jean-Pierre Lepeltier, qui a été Prof ici pendant de longues années
- Du succès grandissant de l'équipe de proba-stat, qui de balbutiante est devenue une équipe qui compte dans le monde des probas
- Merci à tous ceux qui dans la salle contribuent à maintenir cette recherche de tout premier plan, théorique et appliquée

Longue vie au Laboratoire de Math's

- Pas toujours facile d' être aussi performant dans la durée
- A vous tous, (ENSEIGNANTS)-CHERCHEURS , en Mathématiques pures, Appliquées, IA-tisées
- A vous JEUNES DOCTORANTS-CHERCHEURS des différentes spécialités. Nous comptons sur votre enthousiasme maintenir le Cap de l'excellence
- A tous les PERSONNELS, qui vous ont déjà ou vont vous seconder.
- Je vous dis merci...et allez-y, foncez, en groupe et solidaire...

Level sets family

and applications in deterministic and stochastic envelope problems.

from the PhD thesis of Alexandre Prevot

Kernels indicators on $\mathbb{R}$

- $i(u, v) = \mathbf{1}_{\{u \leq v\}}$ and $i(u, v) = \nVdash_{\{u<v\}} = i(u, v\text{-})$
- $v \to i(u, v) = \mathbf{1}_{\{u \leq v\}} = i_u(v)$ is a rc-nondecreasing function
- $\{u < v\}$ is the strict level set of the identity function $v \to v$

Set parameters of A non-empty subset of $\mathbb{R}$, $(\inf\{\emptyset\} = +\infty)$

- The first point of A is $\mathrm{d}_A = \inf\{x \,|\, x \in A\}$.
- The last point of A is $\mathrm{D}_A = \sup\{x \,|\, x \in A\}$.
- The subset A is included in the interval $[\mathrm{d}_A, \mathrm{D}_A]$

Characterization of monotone indicators functions

- $\mathbf{1}_A$ is rc-nondecreasing iff D_A is infinite, d_A is finite and $\mathrm{d}_A \in A$.
- $\mathbf{1}_A(v) = i(\mathrm{d}_A, v)$ and its left-regularized is $\mathbf{1}_A(v\text{-}) = i(\mathrm{d}_A, v)$.
- A is a large level set with boundary d_A

The set $\mathcal{A}_{\mathrm{rc}}$ of right-continuous nondecreasing functions

- its rc-level set is the bivariate set $A_\phi = \{(\lambda, x)\,|\,\lambda \geq \phi(x)\}$

- As x-function, $\mathbf{1}_{A_\phi}(\lambda, x)$ is rc-nondecreasing, and as λ function *lc*-nonincreasing

$\mathcal{A}_{\mathrm{rc}}$-inverse function

- let $\mathrm{d}_\phi^\lambda = \inf\{x\,|\,\lambda \leq \phi(x)\}$ be the first point (in x) of A_ϕ.
- d_ϕ^λ is lc-nondecreasing in λ, so denoted
  $\psi(\lambda\text{-}) = \inf\{x\,|\,\phi(x) \geq \lambda\}$ $\qquad \phi(x) = \sup\{\lambda | \psi(\lambda\text{-}) \leq x\}$
- $\psi(\lambda)$ is the rc-nondecreasing inverse, also denoted $\phi^{\dashv}(\lambda)$

$\mathcal{A}_{\mathrm{rc}}$-inverse function of monotone family

Main properties of strict level sets

- $\{x, \sup_\alpha \phi_\alpha(x) > \lambda\} = \cup_\alpha\{\phi_\alpha(x) > \lambda\}$
- if $\phi_\alpha(x)$ are nondecreasing, $\{\sup_\alpha \phi_\alpha(x) > \lambda\} = \cup_\alpha\{x > \psi_\alpha(\lambda)\}$
- $\{\sup_\alpha \phi_\alpha(x) > \lambda\} = \{x > \inf_\alpha \psi_\alpha(\lambda)\}$

Application to envelope family

Envelope definition

- Let $\mathcal{E}$ be a family stable by inf of functions or processes
- For example, $\mathcal{E}$ is nonincreasing functions or D-supermartingales
- $y(x)$ is a ladlag function, or $Y(t)$ a ladlagD-adapted process

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The E-envelope of y is the infimum of E-functions g larger then y
$\mathcal{E}(y) = \inf\{g \in \mathcal{E} | g \geq y\}$
The Snell-envelope of Y is the essential-inf of Strong Supermartingales Z dominating Y, $R(Y) = \inf\{Z | Z \geq Y\}$, that is for any stopping time, $R(Y)_T = \operatorname{essinf} Z_T | Z_T \geq Y_T\}$

Let $(Y, Z \geq 0)$ and $A_\lambda = \{Y > \lambda R(Y)\}$, $\lambda \in (0,1)$. Then

- $R(R(Y)\mathbf{1}_{A_\lambda}) = R(Y)$, and if $D_T^\lambda = \inf\{s \geq T, Y_s > \lambda R(Y)_s\}$
- $R(Y)_T = \mathbb{E}[R(Y)_{D_T^\lambda} | \mathcal{F}_T]$ and $\lambda R(Y)_{D_T^\lambda} \leq (\sup(Y, Y^+))_{D_T^\lambda}$
- The sequence D_T^λ is nondecreasing, and its limit is the optimal stopping times

Zero bias transformation

Gaussian characterization

- Z real random variable, $\mathbb{E}(Z) = 0, \mathrm{var}(Z) = \sigma^2$
- Stein $Z \sim N(0, \sigma^2)$ iff $\mathbb{E}[Zf(Z)] = \sigma^2 \mathbb{E}[f'(Z)]$

Zero-bias transformation X^*

- Let X, with $\mathbb{E}(X) = 0, \mathrm{var}(X) = \sigma^2$
- Survival nonincreasing probability $\bar{\mu}_X(x) = \mathbb{P}(X > x)$
- New $p^*_X(x)$ is a probability density of some r.v. X^*, where $p^*_X(x) = \mathbb{E}[X\mathbf{1}_{\{X>x\}}]/\sigma^2 = 1/\sigma^2(\mathbb{E}[(X-x)^+] + x\bar{\mu}_X(x))$
- $\mathbb{E}[Xf(X)] = \sigma^2 \mathbb{E}[f'(X^*)]$

.

- Let $X \in \{-p, q\}, p + q = 1$, with $(\mathbb{P}(X = q) = p, \mathbb{P}(X = -p) = q)$, and $\mathbb{E}(X) = 0$, $\mathrm{var}(Z) = \sigma^2$
- Then $X^* \sim U[-p, q]$.

Good exercise for student

Normal approximation

Stein equation for a given function h "derivable"

- Put $\Phi_\sigma(h) = (Z)]$ where $Z \sim N(0, \sigma^2)$
- Solve in f_h the differential equation: $x f_h(x) - \sigma^2 f_h'(x) = h(x) - \Phi_\sigma(h)$.
- Then, taking the expectation we have
$$\left|\mathbb{E}[h(X)] - \Phi_\sigma(h)\right| = \left|\sigma^2 \mathbb{E}\left[f_h'(X^*) - f_h'(X)\right]\right|$$
- $\mathbb{E}[Xf(X)] = \sigma^2 \mathbb{E}[f'(X^*)], \left|\mathbb{E}[h(X)] - \Phi_\sigma(h)\right| \leq \sigma^2 \|f_h''\| \mathbb{E}\left[|X^* - X|\right]$

Sum of independente r.v. X_i, $W = X_1 + \cdots + X_n$ $\sigma_W^2 = \sigma_1^2 + \cdots + \sigma_n^2$

- Let I s.t. $\mathbb{P}(I = i) = \sigma_i^2 / \sigma_W^2$, $W^{(I)} = W - X_I$
- $W^* = W^{(I)} + X_I^*$ is the zero-bias transformate of W.
- $\mathbb{E}\left[|W^* - W|^k\right] = \left(\sim O\left(\frac{1}{\sqrt{n}^k}\right)\right)$

Allow to control the speed of convergence in the central limit theorem.

Forward recovery bi-revealed utility problem

without optimization and control problem.

A regular utility function u is a

- (concave, increasing, positive) function on $[0,\infty)$, with $u(0)=0$.
- Inada condition on its derivative u_z: $u_z(+\infty)=0, u_z(0)=\infty$.

The convex decreasing conjugate utility $\tilde{u}$

- $\tilde{u}(y)=\sup_{x>0}\big(u(x)-x\,y\big)$, with $u'_x(x^*)=y$,
- $\tilde{u}(y)=u(-\tilde{u}_y(y))+y\tilde{u}_y(y), \quad u(x)=x\,u_x(x)-\tilde{u}(u_x(x))$
- Legendre inequality: $\tilde{u}(y)\ -u(x)+x\,y\ \geq 0\ \forall(x,y)>0$

A dynamic utility on $(\Omega,\mathbb{P},(\mathcal{F}_t))$

- is a optional family of randomfield $\mathbf{U}=\{U(t,z),z\in\mathbb{R}^+\}$
- such that $\forall t,(z\to U(t,z))$ is a standard utility function.
- Its conjugate is the field $\widetilde{\mathbf{U}}=\{\widetilde{U}(t,y),y\in\mathbb{R}^+\}$

Concave optimization with constraint

A Backward toy model in Economics

- Data: utility concave function u, its conjugate $\tilde{u}$, a horizon T , $\mathscr{X}$ a convex family of r.v. X_T,
- $\{Y_t \geq 0\}$ a state price process with $\mathbb{E}(X_T Y_T) \leq X_0 Y_0, \forall X_T \in \mathscr{X}\}$.
- *Optimization problem* $max\{\mathbb{E}(u(X_T))|X_T \in \mathscr{X}\}$, with budget constraint $\mathbb{E}(Y_T X_T) \leq x$.

Solution via Lagrange multiplier and conjugate

- Equivalent Pb: $\max\{\mathbb{E}(u(X_T) + y(x - Y_T X_T)|X_T \in \mathscr{X}\} = U_0(x, y)$
- By Fenchel, If $-\tilde{u}_y(yY_T) \in \mathscr{X}$, then optimum is $X_T^* = -\tilde{u}_y(yY_T)$
- y is selected by achieved the budget constaint $\mathbb{E}[-\tilde{u}_y(yY_T)Y_T] = x$
- $\tilde{U}_0(y) = \mathbb{E}[\tilde{u}(yY_T]$ decreasing convex conjugate utility in y.
- Its conjugate at 0, $U_0(x) = \mathbb{E}[u(X_T^*(x)]$ is an increasing concave function.

To recover $\{U(t,z)\}$, the observable also must depend on z

The data

- An initial condition $U(0,z) = u(z)$ a given utility function
- An observed (data) positive adapted random field, $\mathbf{X} = \{X_t(x)\}$,
- The optimal adequation between $\{U(t,z)\}$ and $\{X_t(x)\}$ is the requirement that $\{U(t, X_t(x))\}$ is a martingale for any x.

To limit the size of family of dynamic utilities coherent with the data $\mathbf{X}$, we add constraints on the conjugate utility $\{\tilde{u}(t,y)\}$ via an other family.

- $\mathbf{Y} = \{Y_t(y)\}$ of positive adjoint process,
- Orthogonal to $\{X_t(x)\}$ s.t $X_t(x)Y_t(y)$ is a supermartingale $\forall (x,y)$
- Coherent-optimal to $\tilde{u}(t,y)$, that is $\widetilde{U}(t, Y_t(y))$ is a martingale.
- Bi-revealed problem

Definition Gap function (G.Charlier (2018))

- Let $(u, \tilde{u})$ be a pair of conjugate utility functions $(\mathbb{R}^+ \to \mathbb{R}^+)$
- The bivariate Gap function, $G_u(x, y) = \tilde{u}(y) - u(x) + x\,y \geq 0$
- Fenchel: $G_u(x, u_x(x)) = 0, \forall x > 0$

Bi-revealed utilities at any stopping times τ

- Taking the expectation in $\mathrm{Gap}_U(\tau, X_\tau(x), Y_\tau(y)) = \widetilde{U}(\tau, Y_\tau(y)) - U(\tau, X_\tau(x)) + X_\tau(x) Y_\tau(y) \geq 0$
- Since $\mathbb{E}[\widetilde{U}(\tau, Y_\tau(y))] = y$, $\mathbb{E}[U(\tau, X_\tau(x))] = x$, $\mathbb{E}[X_\tau(x) Y_\tau(y)] \leq xy$

$$\boxed{0 \leq \mathbb{E}[\mathrm{Gap}_U(\tau, X_\tau(x), Y_\tau(y))] \leq \tilde{u}(y) - u(x) + x.y = \mathrm{Gap}_u(x, y)}$$

- $Gap_U(t, X_t(x), Y_t(y))$ is a nonnegative supermartingale, with null expectation if $\mathrm{Gap}_u(x, y) = 0, y = u_x(x)$.
- First order condition $\mathrm{Gap}_U(\tau, X_\tau(x), Y_\tau(u_x(x))) = 0, \mathbb{P}.a, s$ and $Y_t(u_z(x)) = U_z(t, X_t(x))$

Bi-revealed utility and Intrinsic Universe. ++

The utility **U** is bi-revealed by the triplet (u, X, Y) iff (approx)

- $\forall (x,y)$, $\{X_t(x)Y_t(y)\}$ is a supermartingale
- $\{X_t(x)Y_t(u_z(x))\}$ is a martingale
- First order $U_z(t, X_t(x)) = Y_t(u_z(x))$

Asymptotic behavior and intrinsic behavior

- Limit Conditions
  - $\lim_{x\to 0} \frac{X_t(x)}{x} = \Lambda_t^X > 0$ and $\lim_{y\to\infty} \frac{Y_t(y)}{y} = H_t^Y > 0$,
  - $L_t^{\mathrm{int}} = \Lambda_t^X H_t^Y$, $L_0^{\mathrm{int}} = 1$ is a $\mathbb{P}$-martingale.
- The intrinsic universe:$d\mathbb{Q}^{\mathrm{int}} = L_T^{\mathrm{int}}.d\mathbb{P}$, and $X_t^{\mathrm{int}}(x) = X_t(x)/\Lambda_t^X$, $Y_t^{\mathrm{int}}(y) = Y_t(y)/H_t^Y$
- $\mathbb{Q}^{\mathrm{int}}$- Supermartingale properties for $\{X_t^{\mathrm{int}}(x)\}$, $\{Y_t^{\mathrm{int}}(y)\}$, $\{X_t^{\mathrm{int}}(x)\, Y_t^{\mathrm{int}}(y)\}$ and $\{U(t,z)\}$.

Pathwise Utility construction

Hyp $x \to X_t(x)$ is increasing in x with range $[0, \infty)$,

- $U_z(t,z) = Y_t(u_z(X_t^{-1}(z)))$, $\quad U(t,x) = \int_0^x Y_t(u_z(X_t^{-1}(z)))\,dz$
- Why $U(t, X_t(x)) = \int_0^x Y_t(u_x(z))\,d_z X_t(z)$ is a "martingale"
  - Stieljes integral, with explosion near to $z = 0$
  - One to one algebraic bijection between (u, X, U) and (u, X, Y)

Exemple of Differentiable characteristic process

- If $x \mapsto X_t(x)$ to be x-differentiable with $X_x(t,x)$.
- Then, $U(t, X_t(x)) = \int_{(0,x]} Y_t(u_x(z))\,X_x(t,z)dz$
- Under add "regularity", U-martingale is equivalent to $\{Y_t(u_x(x))X_x(t,x)\}$ is martingale
- Then, $\{Y_t(u_x(x))X_x(t,x)\}, \{Y_t(u_x(x))X_t(x)\}$ are martingales

Revealed Utility and Value function

The cone of orthogonal processes Z

- Let Z_t be the family of optional process s.t. $Z_t Y_t(u_x(z_0)$ is supermartingale
- By the properties of the Gap function,
  $0 \leq \mathbb{E}(\mathrm{Gap}(Z_t, Y_t(u_x(z_0) =_{u_x(z_0} -\mathbb{E}(U(t, Z_t) + \mathbb{E}(Z_t Y_t(u_x(z_0)$.
  Then $0 \leq \mathbb{E}(\mathrm{Gap}(Z_t, Y_t(u_x(z_0) \leq u(z_0) - \mathbb{E}(U(t, Z_t)$
- $\mathbb{E}(U(t, X_t(z_0))) = \sup\{E(U(t, Z_t)|Z \text{orthogonal to} Y_t(u_x(z_0)\}$

Application to Itô's framework

- $X_t(x)$, $Y_t(x)$ solution of SDE with regular coefficient to justify the monotony
- $U(t, z)$ dynamic concave random field, satisfying a complex Stochastic PDE, whose the first derivative is given explicitly from the both SDE's
- The complexity of SPDE is strongly reduced