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Asymptotic expansions for functionals of a fractional Brownian motion

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Workshops:
Statistique Asymptotique des Processus Stochastiques

S.A.P.S I - (27 - 28 janvier 1997) - S.A.P.S X (17-20 Mars 2015)

https://lmm.univ-lemans.fr/fr/seminaires-conferences/archives/workshop.html

Workshops: Statistique Asymptotique des Processus Stochastiques

Figure 1: S.A.P.S I - (27 - 28 janvier 1997)

Figure 2: S.A.P.S II (7 et 8 décembre 1998)

Workshops: Statistique Asymptotique des Processus Stochastiques

Figure 3: S.A.P.S III (7 et 8 Décembre 2000)

Data missing ? S.A.P.S IV (19 et 20 Décembre 2002)

Workshops: Statistique Asymptotique des Processus Stochastiques

Figure 4: S.A.P.S V (6-8 Janvier 2005)

Figure 5: S.A.P.S VI (21-23 Mars 2007)

Workshops: Statistique Asymptotique des Processus Stochastiques

Figure 6: S.A.P.S. VII (16-19 Mars 2009)

Figure 7: S.A.P.S VIII (21-24 Mars 2011))

Workshops: Statistique Asymptotique des Processus Stochastiques

Figure 8: S.A.P.S IX (11-14 Mars 2013)

Figure 9: S.A.P.S X (17-20 Mars 2015)

Workshops: Statistique Asymptotique des Processus Stochastiques

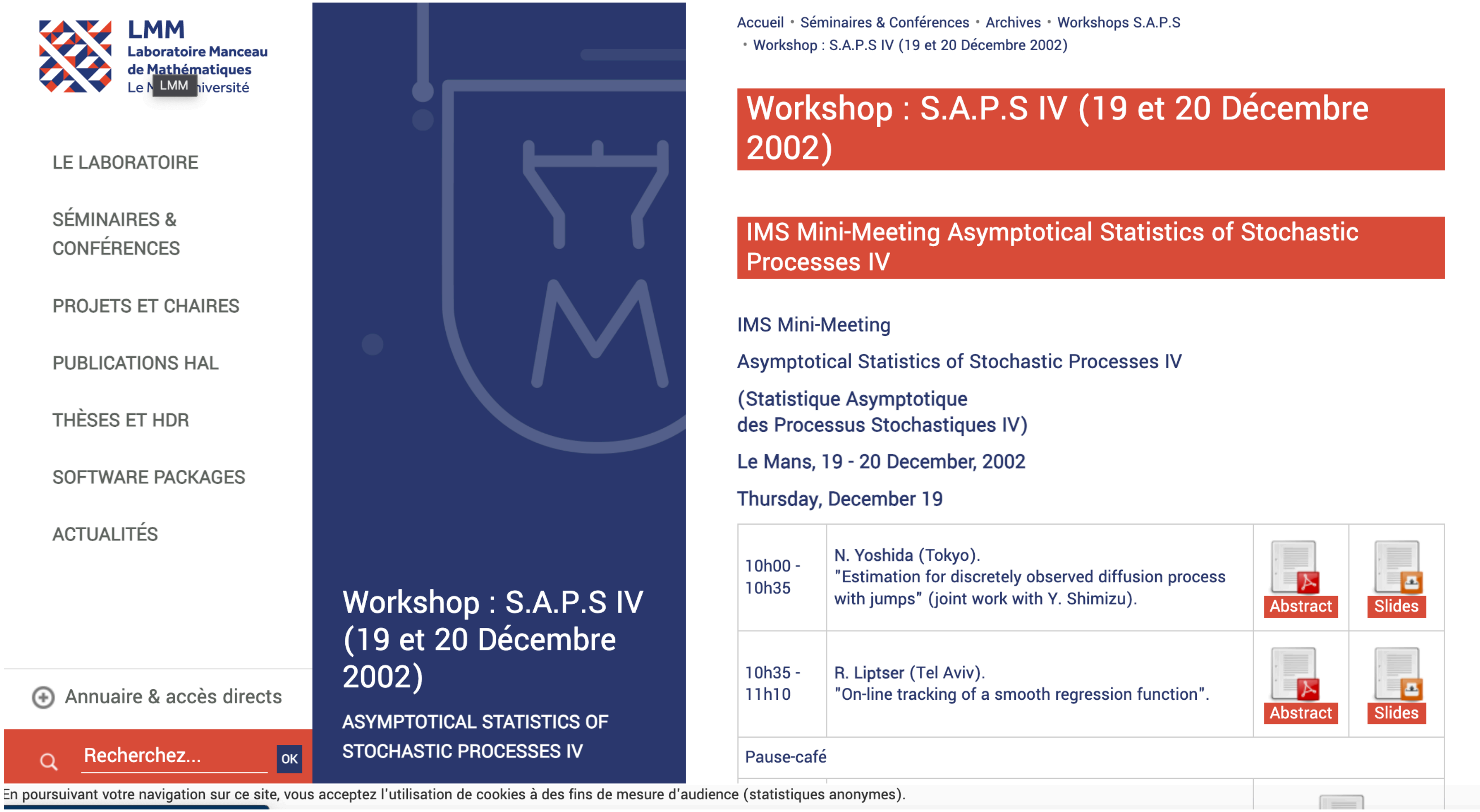

Figure 10: S.A.P.S IV (19 et 20 Décembre 2002) web page

Problem: Where's Yury?

Where's Yury?

Figure 11: S.A.P.S I - (27 - 28 janvier 1997)

Where's Yury?

Figure 12: S.A.P.S II (7 et 8 décembre 1998)

Where's Yury?

Figure 13: S.A.P.S III (7 et 8 Décembre 2000)

Where's Yury?

Figure 14: S.A.P.S V (6-8 Janvier 2005)

Where's Yury?

Figure 15: S.A.P.S VI (21-23 Mars 2007)

Where's Yury?

Figure 16: S.A.P.S. VII (16-19 Mars 2009)

Where's Yury?

Figure 17: S.A.P.S VIII (21-24 Mars 2011)

Where's Yury?

Figure 18: S.A.P.S IX (11-14 Mars 2013)

Where's Yury?

Figure 19: S.A.P.S X (17-20 Mars 2015)

Where's Yury?

Theorem 1. (Conjecture)

Yury has a long memory.

Asymptotic expansions for functionals of a fractional Brownian motion

The fractional Ornstein-Uhlenbeck process

- a fractional Ornstein-Uhlenbeck process

$$\begin{cases} dX_t &= -\theta X_t dt + \sigma dB_t, \quad t \geq 0, \\ X_0 &= x_0, \end{cases} \tag{1}$$

  - x_0: a constant
  - $(B_t, t \geq 0)$: a fractional Brownian motion with Hurst index $H \in (1/2, 1)$.
  - θ unknown, in Θ: a bounded open set in $\mathbb{R}$, $\overline{\Theta} \subset (0, \infty)$,
  - H, σ: known
  - $(X_t)_{t \in [0,T]}$: data
  - $T \to \infty$

The fractional Ornstein-Uhlenbeck process

- Hu and Nualart (Stat Probab Lett 2010): for the estimator

$$\widetilde{\theta}_T = \left(\frac{1}{\sigma^2 H\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}, \tag{2}$$

  the CLT

$$\sqrt{T}\big(\widetilde{\theta}_T - \theta\big) \to^d N(0, c_0)$$

  as $T \to \infty$, when $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$.

- In the inferential theory, the estimator $\widetilde{\theta}_T$ is regarded as an M-estimator for the estimating equation

$$\widetilde{\psi}_T(\vartheta) = \int_0^T X_t^2 dt - \widetilde{\nu}_T(\vartheta) = 0 \tag{3}$$

  for $\widetilde{\nu}_T(\vartheta) = \mu(\vartheta)T \quad \text{with} \quad \mu(\vartheta) = \sigma^2 H\Gamma(2H)\vartheta^{-2H}$.

- Remark. $\widetilde{\theta}_T$ is an approximately moment estimator but not the exact moment estimator since $\overline{\nu}_T(\theta) := E\big[\int_0^T X_t^2 dt \big] = \widetilde{\nu}_T(\theta) + \overline{b}_T(\theta)$ and $\overline{b}_T(\theta)$ does not vanish though it is of $O(1)$ as $T \to \infty$

- Remark. Brouste and Kleptsyna (SISP2010): CLT for the MLE for $H \in (0, 1)$

The fractional Ornstein-Uhlenbeck process

- **The second-order**

$$\mathsf{q} = \mathsf{q}(H) = \begin{cases} \frac{1}{2} & \left(H \in \left(\frac{1}{2}, \frac{5}{8}\right]\right) \\ -4H + 3 & \left(H \in \left(\frac{5}{8}, \frac{3}{4}\right)\right) \end{cases} \tag{4}$$

- **Second-order bias correction**

$$\widehat{\theta}_T^o = \widetilde{\theta}_T - T^{-\frac{1}{2}-\mathsf{q}} \beta(\widetilde{\theta}_T),$$

where $\beta = \beta_H \in C_B^\infty(\Theta)$, i.e., β is smooth on Θ and all its derivatives are bounded on Θ, and $\mathsf{q} = \mathsf{q}(H)$ is a number define by (10).

- **The value of $\widehat{\theta}_T^o$ can exceed the boundary of Θ , the estimator $\widehat{\theta}_T$ we will consider is more precisely defined as**

$$\widehat{\theta}_T = \begin{cases} \widehat{\theta}_T^o & \text{if } \widetilde{\theta}_T \in \Theta \text{ and } \widehat{\theta}_T^o \in \Theta, \\ \theta_* & \text{otherwise,} \end{cases} \tag{5}$$

where θ_* is a prescribed value in Θ. The choice of the value θ_* will not affect asymptotically in any order of expansion.

Problem

- $H \in (1/2, 3/4)$
- Derive asymptotic expansion:

$$\sup_{g \in \mathcal{E}(a,b)} \left| E\big[g\big(T^{1/2}(\widehat{\theta}_T - \theta)\big)\big] - \int_{\mathbb{R}} g(x) p_{H,T}(x) dx \right| = o(T^{-\mathsf{q}(H)})$$

as $T \to \infty$, for every $a, b > 0$.

- For $a, b > 0$, we denote by $\mathcal{E}(a, b)$ the set of measurable functions $g : \mathbb{R} \to \mathbb{R}$ such that $|g(x)| \leq a(1 + |x|^b)$ for all $x \in \mathbb{R}$.

Method: General asymptotic expansion formula for Wiener functionals

Tudor and Y SPA2023

Malliavin calculus: Calculus on the probability space

- $\mathfrak{H}$: a real separable Hilbert space
- $(\mathbb{W}(h))_{h\in\mathfrak{H}}$: an isonormal Gaussian process on a probability space $(\Omega, \mathcal{F}, P)$, i.e., $\mathbb{W}(h)$ $(h \in \mathfrak{H})$ are centered Gaussian variables such that

$$E[\mathbb{W}(h_1)\mathbb{W}(h_2)] \ = \ \langle h_1, h_2\rangle_{\mathfrak{H}}.$$

- DF: the Malliavin derivative of the functional $F : \Omega \to \mathbb{R}$
  - a derivative of F along $\mathfrak{H}$
- $\delta = D^*$: the divergence operator: for $u \in \mathcal{D}(\delta)$,

$$E[\delta(u)F] = E[\langle u, DF\rangle_{\mathfrak{H}}] \quad (F \in \mathbb{D}_{1,2})$$

- e.g. In a finite-dimension,

$$\int F(x)\delta u(x)\phi(x; 0, I_{\mathsf{p}})dx \ = \ \int \langle DF(x), u(x)\rangle\phi(x; 0, I_{\mathsf{p}})dx$$

  for $F \in C^\infty_\uparrow(\mathbb{R}^{\mathsf{p}})$ and $u = (u^i)_{i=1}^{\mathsf{p}} \in C^\infty_\uparrow(\mathbb{R}^{\mathsf{p}}; \mathbb{R}^{\mathsf{p}})$,
  $\delta u(x) = -\mathrm{div}\ u(x) + \sum_{i=1}^{\mathsf{p}} u^i(x)x^i$
  $\Leftrightarrow$ Stein's identity in decision theory

- Asymptotic expansion
  - $|E[e^{\mathrm{i}uF}]| \lesssim (1 + |u|)^{-k} \Rightarrow$ Regularity of the distribution of F
  - Discovery of formulas

General theory: High-order asymptotic expansion of Wiener functionals

- $F_N = \left(F_N^{(1)}, \ldots, F_N^{(d)}\right)$, $N \in \mathbb{N}$: a sequence of d-dimensional centered random variables.
- Gamma Factors $\Gamma^{(1)}(F) = F$, $\Gamma^{(p)}(F) = \langle D(-L)^{-1}\Gamma^{(p-1)}(F), DF\rangle$ of a functional F, where $L = -\delta D$.

[B] For each $j \in \{2, ..., p+1\}$ and $I_j \in \mathbb{I}^j$, $\mathbb{I} = \{1, ..., d\}$, if $k(I_j) \geq 1$, then for $k \in \{1, ..., k(I_j)\}$, there exist sequences of real numbers $(c(I_j, k))_{k=1,...,k(I_j)}$ and $(\gamma(I_j, k))_{k=1,...,k(I_j)}$ such that the following conditions hold.

(i) $0 < \gamma(I_j, 1) < \cdots < \gamma(I_j, k(I_j)) \leq \mathsf{q}$ (when $k(I_j) \geq 1$).

(ii) For $I_2 \in \mathbb{I}^2$,

$$\mathrm{E}\big[\Gamma_{I_2}^{(2)sym}(F_N)\big] - C_{I_2} = \sum_{k=1}^{k(I_2)} c(I_2, k) N^{-\gamma(I_2,k)} + o(N^{-\mathsf{q}}).$$

(iii) For $j \in \{3, ..., p+1\}$ and $I_j \in \mathbb{I}^j$,

$$\mathrm{E}\big[\Gamma_{I_j}^{(j)}(F_N)\big] = \sum_{k=1}^{k(I_j)} c(I_j, k) N^{-\gamma(I_j,k)} + o(N^{-\mathsf{q}}).$$

(iv) .....

High-order asymptotic expansion of Wiener functionals

Then the density function $\widetilde{f}_{N,p,\mathsf{k},\mathsf{q}}$ is expressed as

$$\begin{aligned}\widetilde{f}_{N,p,\mathsf{k},\mathsf{q}}(x) &= \phi(x;0,C)\\ &+\sum_{j=1}^{\mathsf{k}}\sum_{m_1=2}^{p+1}\cdots\sum_{m_j=2}^{p+1}\sum_{I_{m_1}^{(1)}\in\mathbb{I}^{m_1}}\cdots\sum_{I_{m_j}^{(j)}\in\mathbb{I}^{m_j}}\sum_{k_1=1}^{k(I_{m_1}^{(1)})}\cdots\sum_{k_j=1}^{k(I_{m_j}^{(j)})}\Bigg\{\frac{1}{j!m_1\cdots m_j}\\ &\times c(I_{m_1}^{(1)},k_1)\cdots c(I_{m_j}^{(j)},k_j)\, H_{\alpha(I_{m_1}^{(1)},\ldots,I_{m_j}^{(j)})}(x;C)\,\phi(x;0,C)\\ &\times 1_{\{\gamma(I_{m_1}^{(1)},k_1)+\cdots+\gamma(I_{m_j}^{(j)},k_j)\le\mathsf{q}\}}N^{-\{\gamma(I_{m_1}^{(1)},k_1)+\cdots+\gamma(I_{m_j}^{(j)},k_j)\}}\Bigg\}. \end{aligned} \tag{6}$$

- The higher-order asymptotic expansions for a functional F_N is characterized by the expansion of scaled $E[\Gamma^{(p)}(F_N)]$. (Tudor and Y SPA2023)

Theorem 2. Suppose that Conditions $[A1]$, $[A2]$ and $[B]$ are fulfilled. Then

$$\sup_{g\in\mathcal{E}(a,b)}\left|\mathrm{E}\big[g(F_N)\big]-\int_{\mathbb{R}^d}g(x)\widetilde{f}_{N,p,\mathsf{k},\mathsf{q}}(x)dx\right| = o(N^{-\mathsf{q}}) \tag{7}$$

as $N\to\infty$ for every $a,b>0$.

- Distributional expansion. Functionally, near to Watanabe's scheme
  - Non-degeneracy of C is sufficient for the non-degeneracy issue.

High-order asymptotic expansion of Wiener functionals

- **No Markov property**
- **No Cramér's condition**

Get back to the fractional Ornstein-Uhlenbeck process

Tudor and Y (arXiv 2024)

The fractional Ornstein-Uhlenbeck process

- a fractional Ornstein-Uhlenbeck process

$$\begin{cases} dX_t &= -\theta X_t dt + \sigma dB_t, \quad t \geq 0, \\ X_0 &= x_0, \end{cases} \tag{8}$$

  - x_0: a constant
  - $(B_t, t \geq 0)$: a fractional Brownian motion with Hurst index $H \in (1/2, 1)$.
  - θ unknown, in Θ: a bounded open set in $\mathbb{R}$, $\overline{\Theta} \subset (0, \infty)$,
  - H, σ: known
  - $(X_t)_{t\in[0,T]}$: data
  - $T \to \infty$
- Hu and Nualart (Stat Probab Lett 2010)

$$\widetilde{\theta}_T = \left(\frac{1}{\sigma^2 H\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}, \tag{9}$$

The fractional Ornstein-Uhlenbeck process

- **The second-order**

$$\mathsf{q} = \mathsf{q}(H) = \begin{cases} \frac{1}{2} & \left(H \in \left(\frac{1}{2}, \frac{5}{8}\right]\right) \\ -4H + 3 & \left(H \in \left(\frac{5}{8}, \frac{3}{4}\right)\right) \end{cases} \tag{10}$$

- **Second-order bias correction**

$$\widehat{\theta}_T^o = \widetilde{\theta}_T - T^{-\frac{1}{2}-\mathsf{q}} \beta\left(\widetilde{\theta}_T\right),$$

where $\beta = \beta_H \in C_B^\infty(\Theta)$, i.e., β is smooth on Θ and all its derivatives are bounded on Θ, and $\mathsf{q} = \mathsf{q}(H)$ is a number define by (10).

- **The value of $\widehat{\theta}_T^o$ can exceed the boundary of Θ , the estimator $\widehat{\theta}_T$ we will consider is more precisely defined as**

$$\widehat{\theta}_T = \begin{cases} \widehat{\theta}_T^o & \text{if } \widetilde{\theta}_T \in \Theta \text{ and } \widehat{\theta}_T^o \in \Theta, \\ \theta_* & \text{otherwise}, \end{cases} \tag{11}$$

where θ_* is a prescribed value in Θ. The choice of the value θ_* will not affect asymptotically in any order of expansion.

The fractional Ornstein-Uhlenbeck process: strategy

- $\widehat{\theta}_T$ admits a stochastic expansion:

$$T^{1/2}(\widehat{\theta}_T - \theta) = \mathbb{S}_T + T^{-1/2}\kappa\mathbb{S}_T^2 + T^{-\mathfrak{q}}\mathbb{d}_T + \mathrm{R}_T, \tag{12}$$

- $\mathbb{S}_T$ is define by

$$\mathbb{S}_T \;=\; U_T + V_T + W_T, \tag{13}$$

where

$$U_T = I_2(u_T), \quad V_T = I_2(v_T) \;\text{ and }\; W_T = I_1(w_T) \tag{14}$$

for

$$u_T(s,t) \;=\; K_U T^{-1/2} e^{-\theta|s-t|} 1_{[0,T]^2}(s,t) \text{ with } K_U = -\frac{\theta^{2H}}{4H^2\Gamma(2H)},$$

$$v_T(s,t) \;=\; K_V T^{-1/2} e^{-\theta(T-s)-\theta(T-t)} 1_{[0,T]^2}(s,t) \text{ with } K_V = \frac{\theta^{2H}}{4H^2\Gamma(2H)},$$

$$w_T(t) \;=\; K_W T^{-1/2}(e^{-\theta t} - e^{-2\theta T+\theta t}) 1_{[0,T]}(t) \text{ with } K_W = -\frac{x_0\theta^{2H}}{2\sigma H^2\Gamma(2H)}.$$

- $\mathbb{d}_T = T^{-\frac{1}{2}+\mathfrak{q}}\mathrm{G}(\theta)^{-1}\overline{b}_T(\theta) - \beta(\theta)$.
- <u>The expansion (12) is a perturbation of $\mathbb{S}_T$</u>.

The fractional Ornstein-Uhlenbeck process: strategy

- Asymptotic expansion for $\mathbb{S}_T$

$$\sup_{g\in\mathcal{E}(a,b)}\left|E[g(\mathbb{S}_T) - \int_{\mathbb{R}} g(x)p^*_{H,T}(x)dx\right| = o(T^{-\mathsf{q}(H)}) \tag{15}$$

as $T\to\infty$.

- For this, find an expansion of the expected gamma factors $E[\Gamma^{(m)}(\mathbb{S}_T,...,\mathbb{S}_T)]$.
- This is reduced to the expansion of $E\big[\Gamma^{(m)}(U_T^{(1)},...,U_T^{(m)})\big]$ for $U_T^{(1)},...,U_T^{(m)}\in\{U_T,V_T,W_T\}$, e.g.,

$$E\big[\Gamma^{(2)}(U_T,U_T)\big] = C_U(2,H,\theta)+C''_U(2,H,\theta)T^{4H-3}+o(T^{4H-3})$$

as $T\to\infty$.

- Apply the perturbation method to obtain the asymptotic expansion of the estimator:

$$\sup_{g\in\mathcal{E}(a,b)}\left|E\big[g\big(T^{1/2}(\widehat{\theta}_T-\theta)\big)\big] - \int_{\mathbb{R}} g(x)p_{H,T}(x)dx\right| = o(T^{-\mathsf{q}(H)})$$

as $T\to\infty$. □

$$\mathsf{q}=\mathsf{q}(H) = \begin{cases}\frac{1}{2} & (H\in(\frac{1}{2},\frac{5}{8}]) \\ -4H+3 & (H\in(\frac{5}{8},\frac{3}{4}))\end{cases}$$

The fractional Ornstein-Uhlenbeck process

- The density function

$$\begin{aligned} p_{H,T}(x) &= \phi(x;0,c_0)\Big(1 + 1_{\{H\in[\frac{5}{8},\frac{3}{4})\}}2^{-1}c_2H_2(x;0,c_0)T^{4H-3} \\ &\quad +1_{\{H\in(\frac{1}{2},\frac{5}{8}]\}}3^{-1}c_3H_3(x;0,c_0)T^{-\frac{1}{2}} \\ &\quad +c_1H_1(x;0,c_0)T^{-\mathfrak{q}(H)}\Big), \end{aligned} \tag{16}$$

  where $c_0, \ldots, c_3$ are constants depending on H and θ.

- Asymptotic expansion

  Theorem 3. Suppose that $H \in (1/2, 3/4)$. Then

$$\sup_{g\in\mathcal{E}(a,b)} \left| E\big[g\big(T^{1/2}(\widehat{\theta}_T - \theta)\big)\big] - \int_{\mathbb{R}} g(x)p_{H,T}(x)dx \right| = o(T^{-\mathfrak{q}(H)})$$

  as $T \to \infty$, for every $a, b > 0$.

- For $a, b > 0$, we denote by $\mathcal{E}(a,b)$ the set of measurable functions $g : \mathbb{R} \to \mathbb{R}$ such that $|g(x)| \leq a(1 + |x|^b)$ for all $x \in \mathbb{R}$.

- In the second-order (the first-order asymptotic expansion), the long-memory effect appears from $H = 5/8$; the classical rate $T^{-1/2}$ before that.

Table 1: Estimated exponents of $\boldsymbol{T}$ and [Rank]s

sequence\interval	$(\frac{1}{2},\frac{5}{8})$	$(\frac{5}{8},\frac{7}{12})$	$(\frac{7}{12},\frac{2}{3})$	$(\frac{2}{3},\frac{3}{4})$
0th-order term of $\boldsymbol{E}[\boldsymbol{\Gamma}^{(2)}(\boldsymbol{U_T},\boldsymbol{U_T})]$	0 [1]	0 [1]	0 [1]	0 [1]
1st-order term of $\boldsymbol{E}[\boldsymbol{\Gamma}^{(2)}(\boldsymbol{U_T},\boldsymbol{U_T})]$	$\boldsymbol{4H-3}$ [3]	$\boldsymbol{4H-3}$ [2]	$\boldsymbol{4H-3}$ [2]	$\boldsymbol{4H-3}$ [2]
$\boldsymbol{E}[\boldsymbol{\Gamma}^{(3)}(\mathbb{S}_T,\mathbb{S}_T,\mathbb{S}_T)]$	$-\frac{1}{2}$ [2]	$-\frac{1}{2}$ [3]	$-\frac{1}{2}$ [3]	$\frac{3}{2}(4H-3)$ [3]
$\boldsymbol{E}[\widetilde{\boldsymbol{\Gamma}}^{(3)}(\boldsymbol{U_T},\boldsymbol{U_T},\boldsymbol{U_T})]$	-1	$\boldsymbol{-1}$	$\frac{3}{2}(4H-3)$	$\frac{3}{2}(4H-3)$
$\boldsymbol{E}[\widetilde{\boldsymbol{\Gamma}}^{(3)}(\boldsymbol{U_T},\boldsymbol{U_T},\boldsymbol{V_T})]$	-1	$\boldsymbol{-1}$	$\frac{3}{2}(4H-3)$	$\frac{3}{2}(4H-3)$
$\boldsymbol{E}[\widetilde{\boldsymbol{\Gamma}}^{(3)}(\boldsymbol{U'_T},\boldsymbol{U''_T},\boldsymbol{W_T})]$	-1	$\boldsymbol{-1}$	$\boldsymbol{-1}$	$\boldsymbol{-1}$

Simulation study: $H = 0.55$

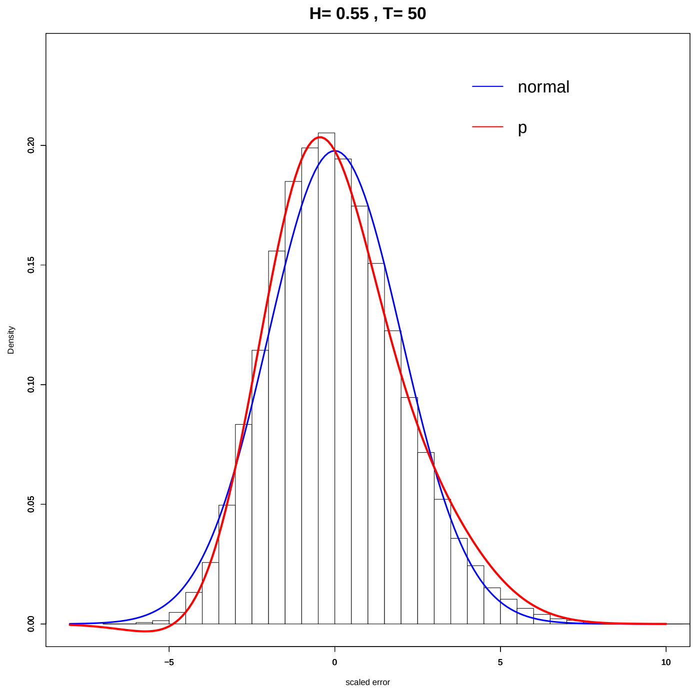

Figure 20: $N(0, c_0)$ and $p_{0.55,50}$

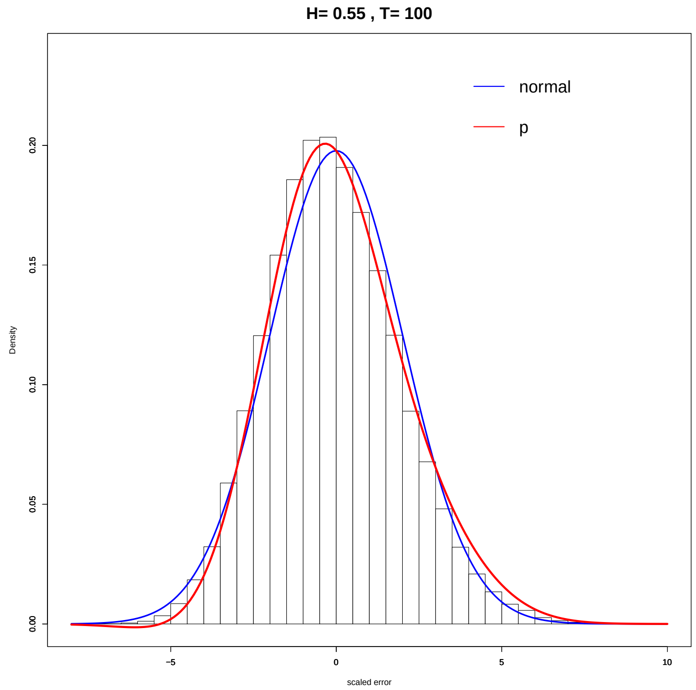

Figure 21: $N(0, c_0)$ and $p_{0.55,100}$

Simulation study: $H = 5/8 = 0.625$

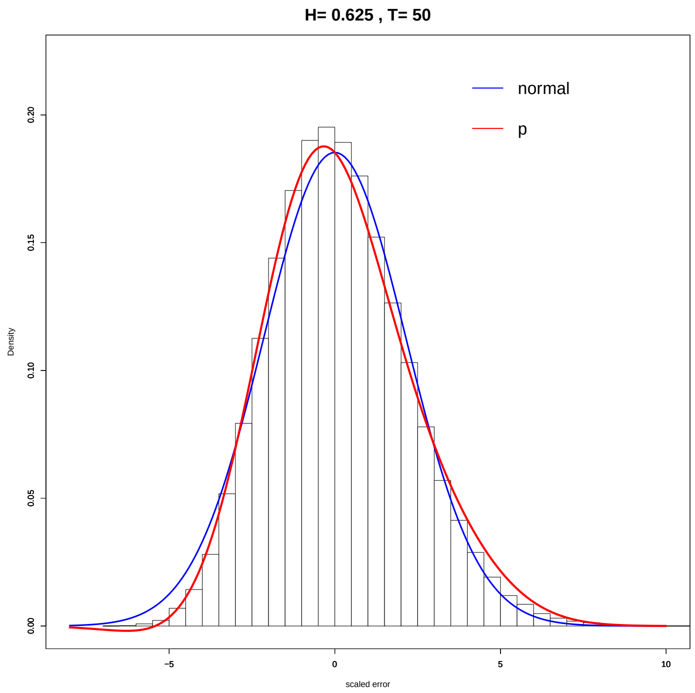

Figure 22: $N(0, c_0)$ and $p_{0.625,50}$

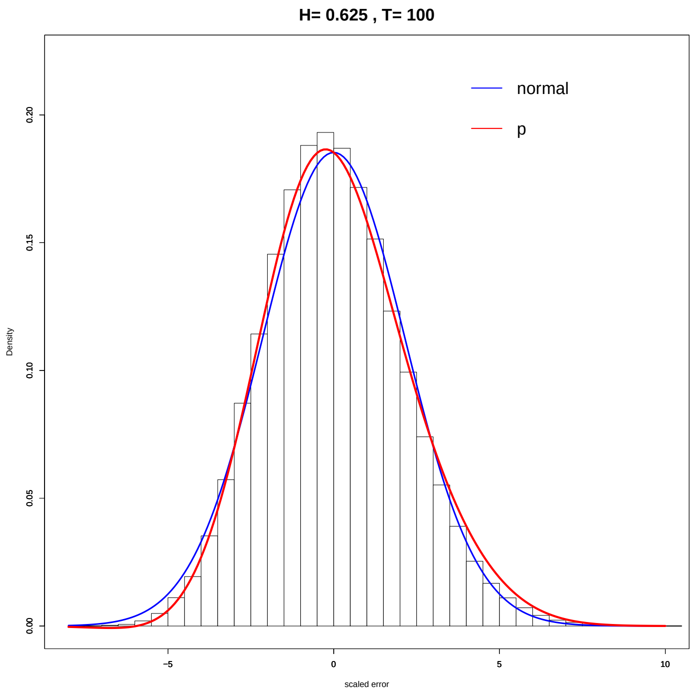

Figure 23: $N(0, c_0)$ and $p_{0.625,100}$

Simulation study: $H = 0.70$

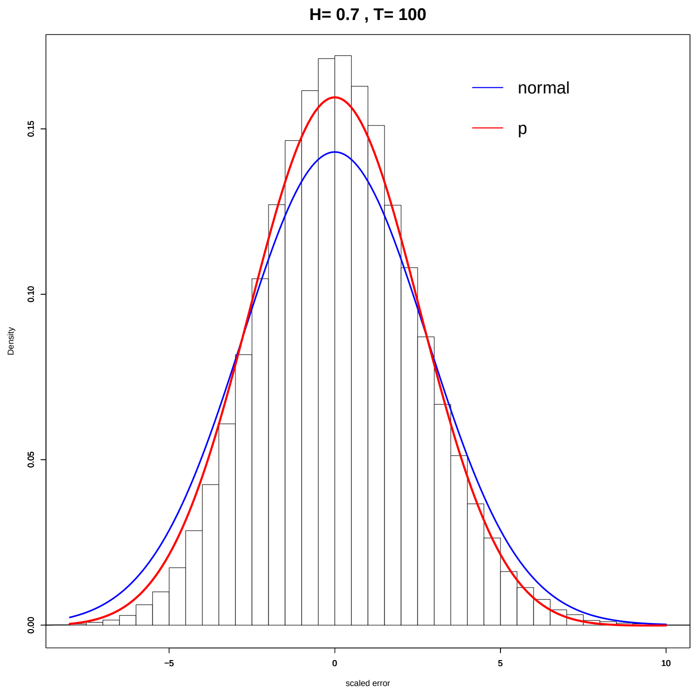

Figure 24: $N(0, c_0)$ and $p_{0.7,100}$

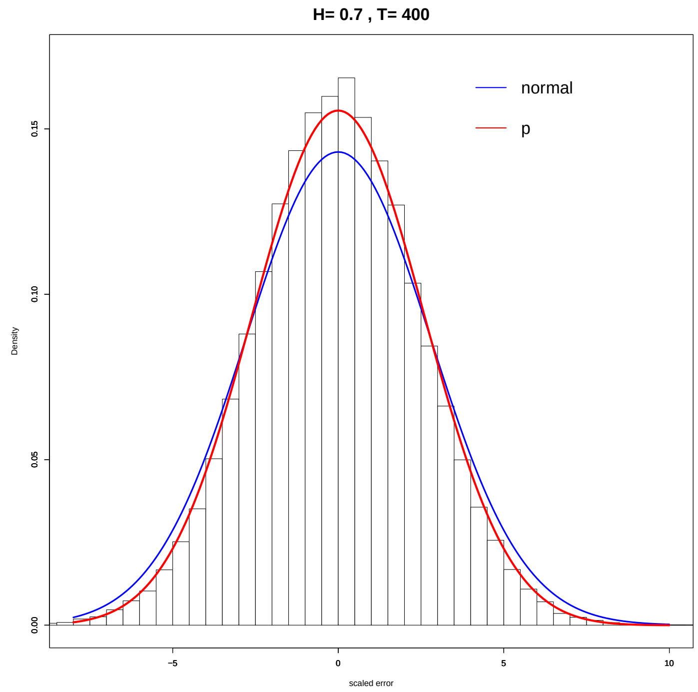

Figure 25: $N(0, c_0)$ and $p_{0.7,400}$

- **(Hu et al.) The limit becomes a normal distribution for $H = 3/4$ with the rate of convergence $T^{1/2}/\sqrt{\log T}$, and a Rosenblatt distribution if H exceeds $3/4$ with the rate T^{2-2H}.**
- **This fact explains the relatively large discrepancy between the histogram and the normal approximation under rate $T^{1/2}$.**

Summary

- **We applied a general asymptotic expansion formula for Wiener functionals, up to any order, in the central limit case.**
  - **The Markovian property is not necessary, in contrast, it is in the traditional theory of asymptotic expansions (the mixing-Markov approach).**
- **Asymptotic expansion was derived for an estimator of the fractional OU process.**

Related topics

- asymptotic expansion of Skorohod integral (Nualart and Y 2019 EJP)
- quadratic variation of a mixed fractional Brownian motion (Tudor and Y 2020 SISP)
- stochastic wave equation (Tudor and Y 2023 SPA)
- Hurst parameter estimation for fBm (Mishura, Yamagishi and Y 2023 SISP)
- variation of Bm with anticipative weights (Y 2023 SPA)
- quadratic variation for a SDE driven by fBm (Yamagishi and Y 2024 SPA)
- Hurst parameter estimation for SDE driven by fBm (Yamagishi 2023)

Congratulations on 30th anniversary of Laboratory of Mathematics !!

30 LMM
Laboratoire Manceau
de Mathématiques
Le Mans Université

with sharing long memories.