Realized cumulants for martingales

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Introduction: realized variance

Let $\Pi_{a,b}$ be a partition of an interval $(a, b] \subset \mathbb{R}$, that is, a finite collection of disjoint subintervals of $(a, b]$ with

$$\bigcup_{(s,t]\in\Pi_{a,b}} (s, t] = (a, b].$$

Denote $|\Pi_{a,b}| = \max_{(s,t]\in\Pi_{a,b}} |t - s|$.
For a process Z, denote $Z_{s,t} = Z_t - Z_s$.
Let M be a (log) price process. The realized variance (a.k.a realized quadratic variation) on a period $(a, b]$ (associated with $\Pi_{a,b}$) is defined as

$$\sum_{(s,t]\in\Pi_{a,b}} |M_{s,t}|^2 .$$

For a sequence of partitions $\Pi^n_{a,b}$ with $|\Pi^n_{a,b}| \to 0$, we have

$$\sum_{(s,t]\in\Pi^n_{a,b}} |M_{s,t}|^2 \to [M]_{a,b} = \langle M^c\rangle_{a,b} + \sum_{t\in(a,b]} |\Delta M_t|^2 \text{ in prob.}$$

Introduction: high-frequency data for estimating low-frequency distribution

If M is an L^2 martingale, then

$$\mathsf{E}[|M_{s,t}|^2] = \mathsf{E}[|M_{0,t}|^2 - |M_{0,s}|^2]$$

and so

$$\mathsf{E}\left[\sum_{(s,t]\in\Pi_{a,b}} |M_{s,t}|^2\right] = \mathsf{E}[|M_{a,b}|^2]$$

that connects high and low frequency distributions.
Neuberger (12) introduced the notion of the aggregation property:

$$\mathsf{E}[g(\mathbb{X}_{s,u})|\mathcal{F}_s] = \mathsf{E}[g(\mathbb{X}_{s,t})|\mathcal{F}_s] + \mathsf{E}[g(\mathbb{X}_{t,u})|\mathcal{F}_s]$$

for $s \le t \le u$. This property is met by $g(x) = x^2$ and $\mathbb{X} = M$.

Introduction: realized skewness

The aggregation property implies

$$\mathsf{E}\left[\sum_{(s,t]\in\Pi_{a,b}} g(\mathbb{X}_{s,t})\right] = \mathsf{E}\left[g(\mathbb{X}_{a,b})\right]$$

for any partition $\Pi_{a,b}$. Neuberger also found that the aggregation property is met by

$$g_2(x,y) = x^3 + 3xy, \quad \mathbb{X}^{(2)} = (M, M^{(2)})$$

if M is an L^3 martingale, where $M_t^{(n)} = \mathsf{E}[(M_T - M_t)^n|\mathcal{F}_t]$ for $t \le T$. Noticing

$$\mathsf{E}[g_2(\mathbb{X}^{(2)}_{0,T})] = \mathsf{E}[(M_T - M_0)^3],$$

Neuberger named $\sum_{(s,t]\in\Pi_{0,T}} g_2(\mathbb{X}^{(2)}_{s,t})$ the realized skewness.

Introduction: realized kurtosis

Bae and Lee (20) extended the idea to find that the aggregation is met by

$$g_3(x, y, z) = x^4 + 6x^2y + 3y^2 + 4xz, \quad \mathbb{X} = (M, M^{(2)}, M^{(3)}).$$

Further,

$$\mathrm{E}[g_3(\mathbb{X}^{(3)}_{0,T})] = \mathrm{E}[(M_T - M_0)^4] - 3\mathrm{E}[(M_T - M_0)^2]^2.$$

Comments:

- the way to find those polynomials was brute-force; actually they showed that there is no other analytic function of $\mathbb{X}^{(3)}$ with the aggregation property (up to a linear combination).
- moments (and cumulants) of an asset under the pricing measure can be computed from its option market data. A bias of the realized cumulant is interpreted as a risk premium.

The main finding in F. and Matsushita (21)

Our finding is that

$$g_n(x_1, \dots, x_n) := B_{n+1}(x_1, \dots, x_n, 0)$$

satisfies the aggregation property with

$$\mathbb{X}^{(n)} = (X^{(1)}, \dots, X^{(n)})$$

for any $n \in \mathbb{N}$, where B_{n+1} is the $(n+1)$-th complete Bell polynomial and $X^{(i)}$ is the ith conditional cumulant process of an L^{n+1} integrable r.v. X. In fact, we show

$$\mathsf{E}[g_n(\mathbb{X}^{(n)}_{s,t})|\mathcal{F}_s] = -\mathsf{E}[X^{(n+1)}_{s,t}|\mathcal{F}_s].$$

When $X = M_T$, then $\mathbb{X}^{(3)} = (M, M^{(2)}, M^{(3)})$, $X^{(n)}_T = 0$ for $n \geq 2$, and

$$X^{(n+1)}_s = \mathsf{E}[g_n(\mathbb{X}^{(n)}_{s,T})|\mathcal{F}_s] = \mathsf{E}\left[\sum_{(t,u]\in\Pi_{s,T}} g_n(\mathbb{X}^{(n)}_{t,u})\middle|\mathcal{F}_s\right].$$

The complete Bell polynomials

Definition.

For $(x_1, \dots, x_n) \in \mathbb{R}^n$, the n th complete Bell polynomial $B_n(x_1, \dots, x_n)$ is defined by

$$B_n(x_1, \dots, x_n) = \frac{\partial^n}{\partial z^n} \exp\left(\sum_{i=1}^n x_i \frac{z^i}{i!} \right) \Bigg|_{z=0}$$

with $B_0 = 1$

Examples are

$$\begin{aligned} B_1(x_1) &= x_1, \\ B_2(x_1, x_2) &= x_1^2 + x_2, \\ B_3(x_1, x_2, x_3) &= x_1^3 + 3x_1 x_2 + x_3, \\ B_4(x_1, x_2, x_3, x_4) &= x_1^4 + 6x_1^2 x_2 + 4x_1 x_3 + 3x_2^2 + x_4, \\ &\dots \end{aligned}$$

$$B_n(x_1, 0, \ldots, 0) = \frac{\partial^n}{\partial z^n} \sum_{k=0}^{\infty} \frac{1}{k!} (x_1 z)^k \bigg|_{z=0} = \frac{\partial^n}{\partial z^n} \frac{1}{n!} (x_1 z)^n \bigg|_{z=0} = (x_1)^n,$$

$$B_n(0, \ldots, 0, x_n) = \frac{\partial^n}{\partial z^n} \sum_{k=0}^{\infty} \frac{1}{k!} \left(x_n \frac{z^n}{n!} \right)^k \bigg|_{z=0} = \frac{\partial^n}{\partial z^n} \left(x_n \frac{z^n}{n!} \right) \bigg|_{z=0} = x_n.$$

Proposition. (binomial property)

Let $n \in \mathbb{N}$ and $(x_1, \ldots x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then,

$$B_n(x_1 + y_1, \ldots, x_n + y_n) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}(x_1, \ldots, x_{n-j}) B_j(y_1, \ldots, y_j).$$

In particular, we have a push-out property:

$$\begin{aligned} B_n(x_1, \ldots, x_{n-1}, x_n) &= B_n(x_1, \ldots, x_{n-1}, 0) + B_n(0, \ldots, 0, x_n) \\ &= B_n(x_1, \ldots, x_{n-1}, 0) + x_n. \end{aligned}$$

Cumulants

Let $p \geq 1$ and $X \in L^p$. For $n \leq p$, the n th cumulant κ_n of X is defined by

$$\kappa_n = (-\sqrt{-1})^n \frac{\partial^n}{\partial z^n} \log \mathrm{E}[e^{\sqrt{-1} z X}]\Big|_{z=0}.$$

We have

$$\mathrm{E}[X^n] = B_n(\kappa_1, \dots, \kappa_n) = B_n(\kappa_1, \dots, \kappa_{n-1}, 0) + \kappa_n$$

because

$$\exp\left(\sum_{n=1}^{\infty} x_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \dots, x_n) \frac{z^n}{n!}$$

as long as convergent, and it is convergent for $B_n(x_1, \dots, x_n) = \mathrm{E}[X^n]$ when $X \in L^\infty$ (Note L^∞ is dense in L^p).
Note that the cumulants are uniquely determined by

$$\kappa_n = \mathrm{E}[X^n] - B_n(\kappa_1, \dots, \kappa_{n-1}, 0).$$

Conditional cumulant process

On a filtered probability space, we define the n th conditional cumulant process $X^{(n)} = \{X_t^{(n)}\}$ by $X_t^{(1)} = \mathrm{E}[X|\mathcal{F}_t]$ and

$$X_t^{(n)} = \mathrm{E}[X^n|\mathcal{F}_t] - B_n(X_t^{(1)}, \dots, X_t^{(n-1)}, 0).$$

We can take a cadlag version.

- Taking a regular conditional distribution of X, almost surely,

$$X_t^{(n)} = (-\sqrt{-1})^n \frac{\partial^n}{\partial z^n} \log \mathrm{E}[e^{\sqrt{-1}zX}|\mathcal{F}_t]\Big|_{z=0}.$$

- When X is $\mathcal{F}_T$ measurable, $X_T^{(1)} = X$ and so,

$$X_T^{(n)} = X^n - B_n(X, 0, \dots, 0) = X^n - X^n = 0$$

  for all $n \geq 2$.

Key lemma

Suppose $X \in L^n$ and let $\mathbb{X}^{(n)} = (X^{(1)}, \dots, X^{(n)})$.

Lemma.

For any stopping times $\tau \le \upsilon$,

$$\mathrm{E}[B_n(\mathbb{X}^{(n)}_{\tau,\upsilon})|\mathcal{F}_\tau] = 0$$

Consequences:

- Let $g_n(x_1, \dots, x_n) = B_{n+1}(x_1, \dots, x_n, 0)$. Then,

$$\mathrm{E}[g_n(\mathbb{X}^{(n)}_{\tau,\upsilon})|\mathcal{F}_\tau] + \mathrm{E}[X^{(n+1)}_{\tau,\upsilon}|\mathcal{F}_\tau] = \mathrm{E}[B_{n+1}(\mathbb{X}^{(n+1)}_{\tau,\upsilon})|\mathcal{F}_\tau] = 0$$

  hence the aggregation property:

$$\mathrm{E}[g_n(\mathbb{X}^{(n)}_{\sigma,\tau})|\mathcal{F}_\sigma] + \mathrm{E}[g_n(\mathbb{X}^{(n)}_{\tau,\upsilon})|\mathcal{F}_\sigma] = \mathrm{E}[g_n(\mathbb{X}^{(n)}_{\sigma,\upsilon})|\mathcal{F}_\sigma].$$

- Not only $B_n(\mathbb{X}^{(n)}_t) = \mathrm{E}[X^n|\mathcal{F}_t]$ but also $B_n(\mathbb{X}^{(n)}_{0,t})$ is a martingale.

Proof of Lemma. When $X \in L^\infty$,

$$\frac{1}{\mathrm{E}[e^{zX}|\mathcal{F}_\tau]} = \left(\sum_{n=0}^{\infty} \mathrm{E}[X^n|\mathcal{F}_\tau]\frac{z^n}{n!}\right)^{-1} = \exp\left(-\sum_{n=1}^{\infty} X_\tau^{(n)}\frac{z^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty} B_n(-\mathbb{X}_\tau^{(n)})\frac{z^n}{n!}$$

on a neighborhood of $z = 0$. This implies

$$\left(\sum_{n=0}^{\infty} \mathrm{E}[X^n|\mathcal{F}_\tau]\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n(-\mathbb{X}_\tau^{(n)})\frac{z^n}{n!}\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathrm{E}[X^{n-j}|\mathcal{F}_\tau]B_j(-\mathbb{X}_\tau^{(j)})\frac{z^n}{n!}$$

is equal to 1, and so for $n \geq 1$,

$$0 = \sum_{j=0}^{n}\binom{n}{j}\mathrm{E}[X^{n-j}|\mathcal{F}_\tau]B_j(-\mathbb{X}_\tau^{(j)}) = \mathrm{E}\left[\sum_{j=0}^{n}\binom{n}{j}\mathrm{E}[X^{n-j}|\mathcal{F}_v]B_j(-\mathbb{X}_\tau^{(j)})\middle|\mathcal{F}_\tau\right].$$

The right hand side coincides with $\mathrm{E}[B_n(\mathbb{X}_v^{(n)} - \mathbb{X}_\tau^{(n)})|\mathcal{F}_\tau]$ by the binom. property.

Example: Lévy process (B_n is time-space harmonic)

When $X = L_T$ for a Lévy process L with triplet (μ, σ^2, ν),

$$X_t^{(1)} = L_t + (T - t)\mu,$$
$$X_t^{(2)} = (T - t)\left(\sigma^2 + \int x^2 \nu(\mathrm{d}x)\right),$$
$$X_t^{(n)} = (T - t)\int x^n \nu(\mathrm{d}x) \quad (n \geq 3)$$

and so,

$$X_{0,t}^{(1)} = L_t - L_0 - \mu t,$$
$$X_{0,t}^{(2)} = -t\left(\sigma^2 + \int x^2 \nu(\mathrm{d}x)\right),$$
$$X_{0,t}^{(n)} = -t\int x^n \nu(\mathrm{d}x) \quad (n \geq 3).$$

We have that $B_n(\mathbb{X}_{0,t}^{(n)})$ is a martingale. In particular for $L = W$ (a Brownian motion), $B_n(\mathbb{X}_{0,t}^{(n)}) = t^{n/2}H_n(t^{-1/2}W_t)$, where H_n is the n th Hermite polynomial.

A cumulant recursion formula

Theorem.

Let $p > 2$, $T > 0$ and $X \in L^p$ be $\mathcal{F}_T$ measurable. Then, for any (possibly stochastic) partition $\Pi_{\sigma,T}$ and for any $n \le p-1$,

$$X_\sigma^{(n+1)} = \mathsf{E}\left[\sum_{(\tau,\upsilon]\in\Pi_{\sigma,T}} g_n(\mathbb{X}_{\tau,\upsilon}^{(n)}) \middle| \mathcal{F}_\sigma \right]$$

Proof: Use the aggregation property and the fact that

$$\mathsf{E}[g_n(\mathbb{X}_{\sigma,T}^{(n)})|\mathcal{F}_\sigma] = -\mathsf{E}[X_{\sigma,T}^{(n+1)}|\mathcal{F}_\sigma] = X_\sigma^{(n+1)}.$$

Here we recall that if X is $\mathcal{F}_T$ measurable and $n \ge 1$, then

$$X_T^{(n+1)} = 0.$$

Realized cumulants

If M is a martingale and $X = M_T$, then $X^{(1)} = M$. We name

$$\sum_{(\tau,v]\in\Pi_{\sigma,T}} g_{n-1}(\mathbb{X}^{(n-1)}_{\tau,v})$$

the n th realized cumulant for M associated with the partition $\Pi_{\sigma,T}$ of the period $(\sigma, T]$.

More generally, for an $\mathcal{F}_T$ measurable random variable X,

$$\sum_{(\tau,v]\in\Pi_{\sigma,T}} g_{n-1}(\mathbb{X}^{(n-1)}_{\tau,v})$$

is an unbiased estimator of the conditional cumulant $X^{(n)}_\sigma$.

High-frequency limit

By taking the high-frequency limit $|\Pi_{\sigma,T}| \to 0$, we have the following.

Theorem.

If $X \in \bigcap_{p>1} L^p$ and $\mathcal{F}_T$ measurable, then

$$X_\sigma^{(n+1)} = \mathrm{E}\left[\sum_{s\in(\sigma,T]} g_n(\Delta\mathbb{X}_s^{(n)}) + \frac{1}{2}\sum_{j=1}^{n}\binom{n+1}{j}\left[X^{(n+1-j)}, X^{(j)}\right]^c_{\sigma,T}\middle|\mathcal{F}_\sigma\right]$$

for any $n \in \mathbb{N}$, where $[\cdot,\cdot]^c$ is the continuous part of the quadratic covariation process.

- Continuous case ($\Delta\mathbb{X}^{(n)} = 0$) by Lacoin, Rhodes and Vargas (23), and Friz, Gatheral and Radoičic (22).
- An independent derivation based on signature cumulants by Friz, Hager and Tapia (22).

(Note for the proof) We use the following lemma Itô's formula provides:

Lemma

For any semimartingale Y and any polynomial g with $g(0) = \partial g(0) = 0$, as $|\Pi_{\sigma,T}| \to 0$, we have

$$\sum_{(\tau,\upsilon]\in\Pi_{\sigma,T}} g(Y_{\tau,\upsilon}) \to \frac{1}{2}\sum_{i,j=1}^{d} \partial_i\partial_j g(0)[Y^i, Y^j]^c_{\sigma,T} + \sum_{t\in(\sigma,T]} g(\Delta Y_t).$$

The sum of the quadratic terms contained in $g = g_n = B_{n+1}(\cdot, 0)$ is

$$B_{n+1,2}(x_1,\dots,x_n) := \frac{1}{2}\sum_{j=1}^{n}\binom{n+1}{j} x_{n+1-j}x_j$$

since

$$\exp\left(\sum_{i=1}^{\infty} x_i \frac{z^i}{i!}\right) = 1 + \sum_{i=1}^{\infty} x_i \frac{z^i}{i!} + \frac{1}{2}\left(\sum_{i=1}^{\infty} x_i \frac{z^i}{i!}\right)^2 + \dots.$$

Application: affine stochastic Volterra equations

Let X be a semimartingale with characteristic $(Yb, Yc, Y\nu)$ (for trunction $\chi(z) = z$) with $X_0 = 0$, where

$$Y_t = \mu(t) + \int_0^t \phi(t-s) X_s \mathrm{d}s,$$

$\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $\phi : (0,\infty) \to \mathbb{R}_+$ is L^1, $b \in \mathbb{R}$, $c \geq 0$ and ν is a measure on $\mathbb{R} \setminus \{0\}$ with $\int |z|^k \nu(\mathrm{d}z) < \infty$, $k = 1, \dots, n$.

Abi Jaber (22): The unique weak solution exists and $Y_T \in L^n$ under some conditions. Examples include

- $(Y, 0, Y\delta_1)$... X is a Hawkes process with the intensity

$$\dot{Y}_t = \dot{\mu}(t) + \int_0^t \phi(t-s) \mathrm{d}X_s.$$

- $(0, Y, 0)$... (hyper rough) Heston model.

Cumulant recursion for affine SVE

Theorem. The n th cumulant process of X_T is given by

$$X_t^{(n)} = \int_t^T \psi^{(n)}(T-s)\Xi_t(\mathrm{d}s)$$

for $n \geq 2$, where

$$\Xi_t(s) = \mathrm{E}[Y_s|\mathcal{F}_t] = \xi_0(s) + \int_0^t \psi(s-u)\mathrm{d}(X-Yb)_u,$$

$$\psi^{(n)} = g_{n-1}^{\nu}(1+b\psi, \psi^{(2)} * \psi, \dots, \psi^{(n-1)} * \psi),$$

$$g_{n-1}^{\nu}(x_1,\dots,x_{n-1}) = \sum_{k=2}^{n} \nu_k B_{n,k}(x_1,\dots,x_{n-k+1}),$$

$$\nu_k = c1_{k=2} + \int z^k \nu(\mathrm{d}z)$$

and $B_{n,k}$ are partial Bell polynomials.

Example: (simplified) affine forward variance model

Consider a continuous martingale $\{X_t\}$ satisfying

$$\mathrm{d}\langle X\rangle_t = \xi_t \mathrm{d}t, \quad \mathrm{d}\xi_t^s = k(s-t)\mathrm{d}X_t, \quad \xi_t^s = \mathsf{E}[\xi_s|\mathcal{F}_t],$$

where k is a square-integrable function with $k(t) = 0$ for $t \leq 0$. The important example is

$$k(t) = \nu\gamma t^{\alpha-1} E_{\alpha,\alpha}(-\gamma t^{\alpha}) 1_{(0,\infty)}(t), \quad E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}$$

corresponding to the rough Heston model (and the Heston when $\alpha = 1$).

In this case, the conditional cumulant processes for X_T are

$$X_t^{(1)} = X_t, \quad X_t^{(2)} = \mathsf{E}[\langle X^{(1)}, X^{(1)}\rangle_{t,T}|\mathcal{F}_t] = \int_t^T \xi_t^s \mathrm{d}s$$

By the recursion formula, we have:

Theorem

For any $n \geq 2$, $X_t^{(n)} = \int_t^T k^{(n)}(T-s)\xi_t^s \mathrm{d}s$, where $k^{(n)}$ is recursively defined as

$$k^{(n+1)} = B_{n+1,2}(1, k^{(2)} * k, \dots, k^{(n)} * k)$$

with $k^{(2)} = 1$.

Comments:

- This is for a simplified version of the (rough) Heston model but essentially the same for the full version of it.
- A reformulation of the diamond calculus by Alòs et al. (2020).
- The Edgeworth expansion up to any order then follows for European option prices under the rough Heston model.
- By the Tauberian theorem, $k^{(n)}(T) = O(T^{(n-2)\alpha})$ as $T \downarrow 0$, and so, $T^{-n/2}X_0^{(n)} = O(T^{(n-2)H})$, $H = \alpha - 1/2$, implying "Power-law skew".

Example: (simplified) affine forward intensity model

Consider a purely discontinuous martingale $\{X_t\}$ satisfying

$$\Delta X_t \in \{0, \epsilon\}, \quad \mathrm{d}\langle X\rangle_t = \lambda_t \mathrm{d}t, \quad \mathrm{d}\lambda_t^s = k(s-t)\mathrm{d}X_t, \quad \lambda_t^s = \mathsf{E}[\lambda_s|\mathcal{F}_t]$$

(a compensated Hawkes process) It is known that X converges to the rough Heston as $\epsilon \to 0$. The conditional cumulant processes for X_T are

$$X_t^{(1)} = X_t, \quad X_t^{(2)} = \mathsf{E}[[X^{(1)}, X^{(1)}]_{t,T}|\mathcal{F}_t] = \int_t^T \lambda_t^s \mathrm{d}s$$

and by the recursion formula, we have:

Theorem

For any $n \geq 2$, $X_t^{(n)} = \int_t^T k^{(n)}(T-s)\lambda_t^s \mathrm{d}s$, where $k^{(n)}$ is recursively defined as

$$k^{(n+1)} = \epsilon^{-2} g_n(\epsilon, \epsilon k^{(2)} * k, \dots, \epsilon k^{(n)} * k)$$

with $k^{(2)} = 1$. Notice $k^{(n+1)} = B_{n+1,2}(1, k^{(2)} * k, \dots, k^{(n)} * k) + O(\epsilon)$.