Set Values and Efficiency of Nonzero Sum Games

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Outline

Nash equilibrium

- Consider a nonzero sum game with two players :
  - ◇ controls : $a = (a_1, a_2) \in A = A_1 \times A_2$
  - ◇ utilities : $J(a) = (J_1(a),\ J_2(a))$

- Nash Equilibrium (NE) : $a^* \in A$

$$J_1(a^*) \geq J_1(a_1, a_2^*), \quad J_2(a^*) \geq J_2(a_1^*, a_2), \quad \forall a_1, a_2$$

Ill-posedness

- An example :

$J(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100, 100)$	$(0, 102)$	$(0, 102)$
$a_1 = 1$	$(102, 0)$	$(1, 2)$	$(0, 0)$
$a_1 = 2$	$(102, 0)$	$(0, 0)$	$(3, 1)$

- Multiple NE : (raw) set value $\mathbb{V}_0 := \{(1,2),(3,1)\}$
- Inefficiency : NEs $<$ socially optimal $(100, 100)$
- Instability : A small perturbation of the game may change the efficiency dramatically

Two goals

- Mechanism design : Improve the efficiency by small "investment"
- Dynamic set value :
  - ◇ DPP
  - ◇ PDE approach

Outline

Efficiency

- Recall the example

$J(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100, 100)$	$(0, 102)$	$(0, 102)$
$a_1 = 1$	$(102, 0)$	$(1, 2)$	$(0, 0)$
$a_1 = 2$	$(102, 0)$	$(0, 0)$	$(3, 1)$

- $\text{Efficiency} = \dfrac{\text{best equilibrium}}{\text{socially optimal control}} = \dfrac{3+1}{100+100} = 2\%$
  - ◇ Price of stability (Anshelevich, Dasgupta, et al 2008)
  - ◇ Relatively easy to implement the best equilibrium

Mechanism 1 : κ-implementation

- κ-implementation (Monderer-Tennenholtz 2003) :
  - ◇ A mediator designs a rewarding mechanism $\pi = (\pi_1, \pi_2) \in \Pi_\kappa$

$$\pi_i \geq 0, \quad \pi_1 + \pi_2 \leq \kappa$$

  - ◇ Consider the modified game : $J^\pi(a) := J(a) + \pi(a)$
- The focus of Monderer-Tennenholtz 2003 :
  - ◇ Find minimum κ to induce a desired outcome
  - ◇ $\cdots$
- Our focus : improvement of efficiency by small κ

$$E_\kappa := \sup_{\pi \in \Pi_\kappa} E(\pi)$$

κ-implementation : an example

- $\kappa = 1$, set $\pi(0,1) = (1,0)$ and $\pi(a) = (0,0)$ for all other a

$J^\pi(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100, 100)$	$(1, 102)$	$(0, 102)$
$a_1 = 1$	$(102, 0)$	$(1, 2)$	$(0, 0)$
$a_1 = 2$	$(102, 0)$	$(0, 0)$	$(3, 1)$

  ◇ $E(\pi) = \frac{0+102}{100+100} = 51\%$

- $\kappa = 4$, set $\pi(0,0) = (2,2)$ and $\pi(a) = (0,0)$ for all other a

$J^\pi(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(102, 102)$	$(0, 102)$	$(0, 102)$
$a_1 = 1$	$(102, 0)$	$(1, 2)$	$(0, 0)$
$a_1 = 2$	$(102, 0)$	$(0, 0)$	$(3, 1)$

  ◇ $E(\pi) = 100\%$

The efficiency function

- The efficiency function

$$E_\kappa = \begin{cases} 2\%, & 0 \le \kappa < 1; \\ 51\%, & 1 \le \kappa < 4; \\ 100\%, & \kappa \ge 4 \end{cases}$$

  - ◇ increasing and right continuous
  - ◇ discontinuous with possibly small discontinuity points
  - ◇ practical importance

Win-win-win situation

- Assume the mediator can charge 5% of their income

	Player 1	Player 2	mediator	Efficiency
$\kappa = 0$ $(3, 1)$	2.85	0.95	0.2	2%
$\kappa = 4$ $(100, 100)$	97	97	6	100%

$\diamond$ $97 = 100 \times 95\% + 2, \quad 6 = (100 + 100) \times 5\% - 4$

Further remarks

- Mechanism 2 : taxation
  - ◇ Both rewarding and punishment, more power for the mediator
- $E_\kappa := \sup_{\pi \in \Pi_\kappa} E(\pi)$
  - ◇ Principal-agent problem with multiple agents
  - ◇ The problem is very challenging in continuous time models

Outline

The dynamic model

- Fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and $[0, T]$
- N-players : $\alpha = (\alpha^1, \cdots, \alpha^N)$
- The model : for $i = 1, \cdots, N$,

$$dX_t^\alpha = b(t, X_t^\alpha, \alpha_t)dt + \sigma(t, X_t^\alpha)dB_t;$$
$$J_i(t, x, \alpha) := \boldsymbol{E}^{t,x}\Big[g_i(X_T^\alpha) + \int_t^T f_i(s, X_s^\alpha, \alpha_s^i)ds\Big].$$

◇ Both B and X can be multidimensional

◇ The volatility control case : $\sigma = \sigma(t, x, \alpha)$, is more involved and is an ongoing work.

◇ For simplicity, in this talk we set $d_B = d_X = 1$ and $\sigma \equiv 1$.

The raw set value

- Definition : $\alpha^* \in \mathcal{A}^N$ is a Nash equilibrium at (t, x), denoted as $\alpha^* \in NE(t, x)$, if

$$J_i(t, x, \alpha^*) \geq J_i(t, x, \alpha^{*,-i}, \alpha^i), \quad \forall\ \alpha^i,\ \forall i.$$

- Raw set value :

$$\mathbb{V}_0(t, x) := \Big\{ J(t, x, \alpha^*) : \alpha^* \in NE(t, x) \Big\} \subset \mathbb{R}^N$$

The set value

- For control problem :

$$V_0 := \sup_\alpha J(\alpha) = \lim_{\varepsilon \to 0} J(\alpha^\varepsilon), \quad \text{not} \quad V_0 := J(\alpha^*)$$

- Define $NE_\varepsilon(t, x)$ in obvious sense and then set value :

$$\mathbb{V}(t, x) := \bigcap_{\varepsilon > 0} \mathbb{V}_\varepsilon(t, x) = \lim_{\varepsilon \to 0} \mathbb{V}_\varepsilon(t, x),$$

$$\mathbb{V}_\varepsilon(t, x) := \Big\{ y : |y - J(\alpha^\varepsilon)| \le \varepsilon \text{ for some } \alpha^\varepsilon \in NE_\varepsilon(t, x) \Big\}.$$

◇ $\mathbb{V}$ is always closed

◇ $\mathbb{V} \supset closure(\mathbb{V}_0)$, and the inclusion could be strict

A few remarks

- For control problem ($N = 1$) with value function $v(t,x)$,

$$\mathbb{V}(t,x) = \{v(t,x)\}, \text{ optimal control } \alpha^* \text{ may not exist}$$

- For two person zero-sum game ($N = 2$) with game value $v(t,x)$,

$$\mathbb{V}(t,x) = \{(v(t,x), -v(t,x))\}, \text{ saddle point } \alpha^* \text{ may not exist}$$

- It is a lot easier to obtain α^ε then to obtain α^*. In that case

$$\mathbb{V} \neq \emptyset = \mathbb{V}_0$$

- Buckdahn-Cardaliaguet-Rainer(2004), Frei-dos Reis(2011), Lin (2012)

The dynamic programming principle

- Recall, for control problem,

$$v(0,x) = \sup_{\alpha_{[0,t]}} \boldsymbol{E}^{0,x}\Big[v(t, X_t^{\alpha_{[0,t]}}) + \int_0^t f(\cdots)ds\Big]$$
$$= \boldsymbol{E}^{0,x}\Big[v(t, X_t^{\alpha^*_{[0,t]}}) + \int_0^t f(\cdots)ds\Big].$$

- Expecting DPP for raw set value :

$$\mathbb{V}_0(0,x) = \Big\{ J(t,\psi;0,x,\alpha^*_{[0,t]}) : \text{all } \psi : \mathbb{R} \to \mathbb{R}^N, \alpha^* \in \mathcal{A}^N$$
$$\psi(x') \in \mathbb{V}_0(t,x'), \forall x' \in \mathbb{R}, \quad \alpha^*_{[0,t]} \in NE(t,\psi;0,x) \Big\}$$

◇ Subgame on $[0,t]$ with terminal condition ψ :

$$J_i(t,\psi;0,x,\alpha) := \boldsymbol{E}^{0,x}\Big[\psi_i(X_{t_2}^{\alpha}) + \int_0^t f(\cdots)ds\Big]$$

The admissible controls

- The set value is extremely sensitive to admissible controls !
- Open loop controls $\alpha_i = \alpha_i(t, B_{[0,t]})$: DPP fails !
- State dependent closed loop controls $\alpha_i = \alpha_i(t, X_t)$: DPP fails !
- We have to allow for path dependent controls $\alpha_i(t, X_{[0,t]})$
  - ◇ With $\alpha \in \mathcal{A}^N_{path}$, the (raw) set value is still state dependent, denoted as $\mathbb{V}_{0,path}(t,x)$.
  - ◇ We have truly path dependent α^*, in particular,

$$\mathbb{V}_{0,state}(t,x) \neq \mathbb{V}_{0,path}(t,x).$$

The dynamic programming principle

Theorem (Feinstein-Rudloff-Z. 2020)

$$\mathbb{V}_0(0,x) = \Big\{ J(t,\psi;0,x,\vec{\alpha}^*_{[0,t]}) : \alpha^* \in NE_{path}(t,\psi;0,x)$$
$$\psi : C([0,t]) \to \mathbb{R}^N, \quad \psi(\mathrm{x}_{[0,t]}) \in \mathbb{V}_0(t,\mathrm{x}_t) \Big\}$$

- $J_i(t,\psi;0,x,\vec{\alpha}) := \boldsymbol{E}^{0,x}\Big[\psi(X^{\vec{\alpha}}_{[0,t]}) + \int_0^t f(\cdots)ds\Big]$.
- The DPP holds when b, f, g are also path dependent
- The DPP holds for the set value $\mathbb{V}$, after obvious modification
- Abreu-Pearce-Stacchetti (1990), Sannikov (2007)

Hamiltonian and PDE

- Introduce

$$h(t,x,z,a) := b(t,x,a)z + f(t,x,a) \in \mathbb{R}^N$$

- For control problem ($N = 1$),

$$H(t,x,z) := \sup_a h(t,x,z,a),$$
$$\partial_t v + \frac{1}{2}\partial_{xx} v + H(t,x,\partial_x v) = 0.$$

- For two person zero-sum game ($N = 2$) under Issacs condition :

$$H_1(t,x,z) := \inf_{a_1} \sup_{a_2} h_1(t,x,z,\vec{a}) = \sup_{a_2} \inf_{a_1} h_1(t,x,z,\vec{a}),$$
$$\partial_t v_1 + \frac{1}{2}\partial_{xx} v_1 + H_1(t,x,\partial_x v_1) = 0,$$
$$H_2 = -H_1, \quad v_2 = -v_1$$

The set valued Hamiltonian $\mathbb{H}$

- Fix (t,x,z), the mapping $a \mapsto h(t,x,a)$ is a static game, then we may introduce set valued Hamiltonian $\mathbb{H}(t,x,z) \subset \mathbb{R}^N$ naturally.

- For control problem $(N=1)$, $\mathbb{H}(t,x,z) = \{H(t,x,z)\}$

- For two person zero-sum game $(N=2)$:
  - ◇ under Issacs condition : $\mathbb{H} = \Big\{\Big(H_1, -H_1\Big)\Big\}$
  - ◇ without Issacs condition : $\mathbb{H}(t,x,z) = \emptyset$
  - ◇ Isaacs condition $\Longleftrightarrow \mathbb{H} \neq \emptyset$ $\quad$ ($\mathbb{H}_0$ can be empty)

Vector valued PDE approach for set values (Qiao-Z. 2024+)

• Assume $\mathbb{H}(t,x,z) = \{H(t,x,z)\}$, then $\mathbb{V}(t,x) = \{v^H(t,x)\}$,

$$\partial_t v_i^H + \frac{1}{2}\partial_{xx} v_i^H + H_i(t,x,\partial_x \vec{v}^H) = 0, \quad i = 1,\cdots,N.$$

◇ This covers the control problem and zero sum game problem under Issacs condition

• In the general case, roughly speaking (need the ε-approximations)

$$\mathbb{V}(t,x) = \Big\{ v^H(t,x) : \text{ all } H \text{ s.t. } H(t,x,z) \in \mathbb{H}(t,x,z), \forall (t,x,z) \Big\}$$

• Hamadene-Lepeltier-Peng(1997), Bensoussan-Frehse(2000) : Showed $\supset$ in terms of $\mathbb{V}_0$ and $\mathbb{H}_0$

Set valued PDE approach (Iseri-Z. (???))

$$\sup_{\eta\in\mathcal{T}_{\mathbb{V}},h\in\mathbb{H}(t,x,\partial_x\mathbb{V}(t,x,y)+\eta)} n_{\mathbb{V}}\cdot\Big[\partial_t\mathbb{V}+\frac{1}{2}\partial_{xx}\mathbb{V}+h$$
$$-\mathrm{tr}\,(\eta^\top\partial_x n_{\mathbb{V}}\sigma+\frac{1}{2}\partial_y n_{\mathbb{V}}\ \eta\eta^\top)n_{\mathbb{V}}\Big](t,x,y)=0$$

- y is on the boundary of $\mathbb{V}(t,x)$
- The normal vector $n_{\mathbb{V}}(t,x,y)$ is part of the solution
- η is on the tangent space : $\eta\cdot n_{\mathbb{V}}=0$
- $\partial_t\mathbb{V},\partial_x\mathbb{V},\partial_{xx}\mathbb{V}$ are appropriately defined set valued derivatives
- In terms of $\mathbb{H}_0$, h part means

  $$h(t,x,\partial_x\mathbb{V}(t,x,y)+\eta,\vec{a}^*) \text{ over all equilibria } \vec{a}^*$$
- The key of the theory is a set valued Itô formula (Iseri-Z. 2023)

Thank you very much for your attention !