Path-wise uniqueness for a system of SDE with singular coefficients

30^{th} birthday of LMM, Université Le Mans

Paper: Scaling limits of bisexual Galton-Watson processes (2023)

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First Slide: Problem

$$(S)\begin{cases} X_t &= x_0 + \int_0^t b_1(X_s, Y_s)ds + \int_0^t \sqrt{\ell_1(X_s, Y_s)}\, dB^1_s \\ & \quad + \int_0^t \int_{\mathbb{R}^2_+} \mathbb{1}_{\theta \le \kappa_1(X_{s-}, Y_{s-})} p_1(X_{s-}, Y_{s-}) h(z)\, \widetilde{N}^1(ds, d\theta, dz) \\ & \quad + \int_0^t \int_{\mathbb{R}^2_+} \mathbb{1}_{\theta \le \kappa_1(X_{s-}, Y_{s-})} p_1(X_{s-}, Y_{s-})(z - h(z))\, N^1(ds, d\theta, dz); \\ Y_t &= y_0 + \int_0^t b_2(X_s, Y_s)ds + \int_0^t \sqrt{\ell_2(X_s, Y_s)}\, dB^2_s \\ & \quad + \int_0^t \int_{\mathbb{R}^2_+} \mathbb{1}_{\theta \le \kappa_2(X_{s-}, Y_{s-})} p_2(X_{s-}, Y_{s-}) h(z)\, \widetilde{N}^2(ds, d\theta, dz) \\ & \quad + \int_0^t \int_{\mathbb{R}^2_+} \mathbb{1}_{\theta \le \kappa_2(X_{s-}, Y_{s-})} p_2(X_{s-}, Y_{s-})(z - h(z))\, N^2(ds, d\theta, dz). \end{cases}$$

B^1, B^2 are BM, N^1, N^2 are Poisson point measures on $\mathbb{R}^3_+$ with intensities $ds\, d\theta\, \lambda_i(dz) : i = 1, 2$. $\widetilde{N}^i$ are the compensated Poisson.

Second Slide: Hypotheses

H1) $\int_0^\infty (z^2 \wedge 1)\,\lambda(dz) < \infty$, where $\lambda = \lambda_1 + \lambda_2$.

H2) $b_i, \ell_i, \kappa_i, p_i$ are locally Lipschitz on $\mathbb{R}_+ \times \mathbb{R}_+$.
$b_i(0,z) = \ell_i(0,z) = \kappa_i(0,z) = b_i(z,0) = \ell_i(z,0) = \kappa_i(z,0) = 0$.

H3) ℓ_i, κ_i, p_i are nonnegative, $0 < p_i \leq 1$.

H4) b_i, ℓ_i, κ_i have linear growth and p, q are bounded by 1.

H5) $h \in C_b(\mathbb{R}_+, \mathbb{R}_+)$ and $h(z) = z$ in a neighborhood of 0.

H6) Ellipticity assumption: $\forall\ 0 < \delta \leq n < \infty$
$0 < \zeta(\delta, n) = \inf\{\ell_i(x,y) : (x,y) \in [\delta, n]^2\}$.

H7) $\exists \varepsilon_0 > 0$
$$\liminf_{a \downarrow 0} \left[e^{\varepsilon_0 \int_a^1 z\,\lambda(dz)} \int_0^a z^2 \lambda(dz) \right] = 0.$$

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Third Slide: Main Tools

T1) Burkhölder-Davis-Gundy inequality with $p = 1$:
$\exists 0 < c_1 \leq C_1 < \infty$ for all local martingales M

$$c_1 \mathbb{E}([M,M]_\tau^{1/2}) \leq \mathbb{E}\left(\sup_{s \leq \tau} |M_s|\right) \leq C_1\, \mathbb{E}([M,M]_\tau^{1/2})$$

T2) $\exists 0 < \bar{\mathbf{c}}_1$, if the jumps of M are bounded $\boldsymbol{\Delta}$ then (see Lenglart-Lépingle-Pratelli (1980))

$$\bar{\mathbf{c}}_1 \mathbb{E}(\langle M,M\rangle_\tau^{1/2}) \leq \mathbb{E}\left(\sup_{s \leq \tau} |M_s|\right) + \boldsymbol{\Delta};$$
$$\mathbb{E}([M,M]_\tau^{1/2}) \leq 3\,\mathbb{E}(\langle M,M\rangle_\tau^{1/2}).$$

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Forth Slide: Main Result

Theorem

Under $H1 - H6$ there exists a unique strong solution to the system (S) up to $T_{\mathbf{e}} = \sup_n T_n$, $T_n = \inf\{s : X_s > n \text{ or } Y_s > n\}$.
If $x_0 = 0$ then $X_t = x_0, Y_t = y_0$.

If $\int (z^2 \wedge z)\lambda(dz) < \infty$ then $T_{\mathbf{e}} = \infty$ $\mathbb{P}-$a.s. and for all t

$$\mathbb{E}\left(\sup_{s\leq t} X_s + Y_s\right) \leq Ae^{Ct}$$

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Fifth Slide: Proof

1) We assume that $\int (z^2 \wedge z)\lambda(dz) < \infty$, which we remove by localizing for the big jumps.... Then Grönwall gives for all s $\mathbb{E}\left(X_s + Y_s\right) \leq Ae^{Cs}$ and BDG implies $\mathbb{E}\left(\sup_{s \leq t} X_s + Y_s\right) \leq A' e^{C' t}$.

2) $(X, Y), (\tilde{X}, \tilde{Y})$ two possible solutions

$$T_{\delta,n} = \inf\{t > 0 : X_t \wedge Y_t \wedge \tilde{X}_t \wedge \tilde{Y}_t \leq \delta \quad \text{or} \quad X_t \vee Y_t \vee \tilde{X}_t \vee \tilde{Y}_t \geq n \}...$$

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