Recent results on Epidemic Models

Etienne Pardoux (I2M, AMU)

joint work with R. Forien (INRAE), G. Pang (Rice Univ. USA), A.B. Zotsa (AMU)

Le Mans
In 1927, W.O. Kermack and A.G. McKendrick published a paper which proposes a SIR model of propagation of epidemic diseases with an infection–age dependent infectivity and an infection–age dependent recovery rate. In that paper, they treated also the special case of constant infectivity and constant recovery rate.

That paper was quoted more than 10,000 times, but almost all successors of these pioneers considered only the model with constant rate, which is the following ODE model.

\[
\begin{align*}
\frac{dS}{dt}(t) &= -\lambda S(t) I(t), \\
\frac{dl}{dt}(t) &= \lambda S(t) I(t) - \rho I(t), \\
\frac{dR}{dt}(t) &= \rho I(t).
\end{align*}
\]

As we shall see below, the general model involves an integral equation of Volterra type (or alternatively a PDE).
Kermack and McKendrick, two pioneers

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\frac{dS}{dt}(t) = -\lambda S(t)I(t),
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\]

As we shall see below, the general model involves an integral equation of Volterra type (or alternatively a PDE).
Constant vs. non–constant recovery rate

Let $\mathcal{I}$ denote the length of the infectious period of a typical individual.

- $\mathbb{P}(\mathcal{I} > t) = \exp(-\mu t)$, $t > 0$, means that $\mathcal{I}$ follows the exponential distribution with parameter $\mu$.

- Let $t \mapsto \mu(t)$ be measurable from $\mathbb{R}_+$ into itself, s.t. $\int_0^\infty \mu(t)dt = +\infty$. Assume

\[
(*) \quad \mathbb{P}(\mathcal{I} > t) = \exp \left( - \int_0^t \mu(s)ds \right), \quad t > 0.
\]

It is easy to deduce that $\mathcal{I}$ has the density

\[
f(t) = \mu(t) \exp \left( - \int_0^t \mu(s)ds \right).
\]

- Conversely, given a probability density $f$ on $\mathbb{R}_+$, define its hazard function as $\mu(t) := (\int_t^\infty f(s)ds)^{-1} f(t)$, then we have $(*)$.

- Assuming that the individuals recover at an infection–age dependent rate allows to choose an arbitrary distribution with density for $\mathcal{I}$. 

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30 ans du LMM

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If $P(t)$ is a standard Poisson process, then $P(\lambda t)$ is a rate $\lambda$ Poisson process, and $P(\int_0^t \lambda(s)ds)$ is a rate $\lambda(t)$ Poisson process.

If $Q$ is a standard PRM on $\mathbb{R}^2_+$, i.e. $Q$ is a sum of Dirac measures at random points, the numbers of points in disjoints subsets are mutually independent, and the number of points in $A$ follows the $\text{Poi}(\text{Leb}(A))$ distribution, then

$$\int_0^t \int_0^\infty 1_{u \leq \lambda(s)} Q(ds, du)$$

is a rate $\lambda(t)$ Poisson process.

Law of large numbers : as $N \to \infty$, $N^{-1}P(Nt) \to t$ a.s., locally uniformly in $t$ (standard LLN + Dini theorem).
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Interlude: Poisson process and PRM

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For all $t \geq 0$, $S^N(t) + I^N(t) + R^N(t) = N$.

Suppose that each infectious individual meets others at rate $\beta$. With probability the proportion of susceptibles in the population, i.e. $S^N(t)/N$, the individual who is met is susceptible, in which case the encounter results in a new infection with probability $p$. Let $\lambda := \beta p$. Assume that the durations of the infectious periods of the various individuals are i.i.d., with the law $\text{Exp}(\mu)$.

Then (with $P_{\text{inf}}$ and $P_{\text{rec}}$ two independent standard Poisson proc.)

$$S^N(t) = S^N(0) - P_{\text{inf}}\left(\lambda \int_0^t N^{-1} S^N(s)I^N(s)ds\right),$$

$$I^N(t) = I^N(0) + P_{\text{inf}}\left(\lambda \int_0^t N^{-1} S^N(s)I^N(s)ds\right) - P_{\text{rec}}\left(\mu \int_0^t I^N(s)ds\right),$$

$$R^N(t) = R^N(0) + P_{\text{rec}}\left(\mu \int_0^t I^N(s)ds\right).$$
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\[
S^N(t) = S^N(0) - P_{\text{inf}} \left( \lambda \int_0^t N^{-1} S^N(s) I^N(s) \, ds \right),
\]
\[
I^N(t) = I^N(0) + P_{\text{inf}} \left( \lambda \int_0^t N^{-1} S^N(s) I^N(s) \, ds \right) - P_{\text{rec}} \left( \mu \int_0^t I^N(s) \, ds \right),
\]
\[
R^N(t) = R^N(0) + P_{\text{rec}} \left( \mu \int_0^t I^N(s) \, ds \right).
\]
For all $t \geq 0$, $S^N(t) + I^N(t) + R^N(t) = N$.

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R^N(t) &= R^N(0) + P_{\text{rec}}\left(\mu \int_0^t I^N(s) ds\right).
\end{align*}
Let \((\bar{S}^N(t), \bar{I}^N(t), \bar{R}^N(t)) = N^{-1}(S^N(t), I^N(t), R^N(t))\).

It follows from the LLN for Poisson processes that
\((\bar{S}^N(t), \bar{I}^N(t), \bar{R}^N(t)) \to (\bar{S}(t), \bar{I}(t), \bar{R}(t))\), where

\[
\begin{align*}
\frac{d\bar{S}(t)}{dt} &= -\lambda \bar{S}(t)\bar{I}(t), \\
\frac{d\bar{I}(t)}{dt} &= \lambda \bar{S}(t)\bar{I}(t) - \mu \bar{I}(t), \\
\frac{d\bar{R}(t)}{dt} &= \mu \bar{I}(t).
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It follows from the LLN for Poisson processes that
\[(\bar{S}^N(\tau), \bar{I}^N(\tau), \bar{R}^N(\tau)) \to (\bar{S}(\tau), \bar{I}(\tau), \bar{R}(\tau)),\]
where
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\frac{d\bar{S}(t)}{dt} = -\lambda \bar{S}(t)\bar{I}(t),
\]
\[
\frac{d\bar{I}(t)}{dt} = \lambda \bar{S}(t)\bar{I}(t) - \mu \bar{I}(t),
\]
\[
\frac{d\bar{R}(t)}{dt} = \mu \bar{I}(t).
\]
Let us go back to the general model of Kermack–McKendrick, first with a fixed infectivity $\lambda$, during an infectious period of length $I$, where $P(I \leq t) = F(t)$. The infection age recovery rate is the hazard function of $I$, i.e. $\mu(t) := f(t)/F_c(t)$, if $f$ if the density of $F$, and $F_c(t) := 1 - F(t)$, i.e. $\mathbb{P}(I > t) = \exp(-\int_0^t \mu(s)ds)$.

Then the stochastic model is non-Markov, unless if $F$ is an exponential distribution (i.e. $\mu$ is constant).

$$(\bar{S}^N(t), \bar{I}^N(t), \bar{R}^N(t)) \to (\bar{S}(t), \bar{I}(t), \bar{R}(t)) \text{ as } N \to \infty,$$

where

$$\bar{S}(t) = \bar{S}(0) - \lambda \int_0^t \bar{S}(s)\bar{I}(s)ds,$$

$$\bar{I}(t) = \bar{I}(0)F^c_0(t) + \lambda \int_0^t F^c(t-s)\bar{S}(s)\bar{I}(s)ds,$$

$$\bar{R}(t) = \bar{I}(0)F_0(t) + \lambda \int_0^t F(t-s)\bar{S}(s)\bar{I}(s)ds.$$
SIR model with memory as LLN limit of non–Markov stochastic models

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- $(\bar{S}^N(t), \bar{I}^N(t), \bar{R}^N(t)) \to (\bar{S}(t), \bar{I}(t), \bar{R}(t))$ as $N \to \infty$, where

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  \]

  \[
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Model with constant vs. infection age dependent recovery rate for the Covid

Model with memory vs. Markov model predictions

Hospital deaths

Model with memory

Markov model

Start of lockdown

Daily hospital deaths in France


10 1
100
101
102
103
Extended Data Fig. 1 | Inferred infectiousness profile. Infectiousness was assumed to start from 1 days (top left) to 7 days (bottom right) before symptom onset.
Varying infectivity

Let \( \{ \lambda(t), \ t \geq 0 \} \) be a random function with \( \geq 0 \) values. If

\[
E = \inf\{ t > 0, \lambda(t) > 0 \} \quad I = \sup\{ t > 0, \lambda(E + t) > 0 \}.
\]

Then \( E \) is the exposed period, \( I \) the infectious period.

We assume that to each individual is attached a copy \( \lambda_i(t) \), where the \( \lambda_i \) are i.i.d. To the initially infected individuals are attached copies \( \lambda^0_j(t) \) of another type of infectivity function.

We request that \( \lambda \) belongs a.s. to the Skorohod space \( D \), and that \( \sup_{t \geq 0} \lambda(t) \leq \lambda^* \), where \( \lambda^* \) is a given constant. Then one can establish a law of large numbers as \( N \to \infty \) of the corresponding individual based model. Define the total force of infection

\[
\tilde{F}^N(t) = \sum_{j=1}^{I^N(0)} \lambda^0_j(t) + \sum_{i=1}^{A^N(t)} \lambda_i(t - \tau_i^N) = \lambda \times \# \text{ of infectious indiv.}
\]

\[
A^N(t) = \sum_{i \geq 1} 1_{(0,t]}(\tau_i^N) \text{ counts the number of indiv. infected on } (0, t].
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  \mathcal{E} = \inf\{t > 0, \lambda(t) > 0\} \quad \mathcal{I} = \sup\{t > 0, \lambda(\mathcal{E} + t) > 0\} .
  \]
  Then \( \mathcal{E} \) is the exposed period, \( \mathcal{I} \) the infectious period.

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  \]
Varying infectivity: the LLN

1. Let $S^N(t), I^N(t), R^N(t)$ denote resp. the number of susceptible, infected and recovered indiv. in the population.
2. $S^N(t) + I^N(t) + R^N(t) = N$. Let $\tilde{S}^N = N^{-1}S^N$, same with $I$ and $R$.
3. $(\tilde{S}^N(t), \tilde{S}(t), \tilde{I}(t), \tilde{R}(t)) \rightarrow (\bar{S}(t), \bar{S}(t), \bar{I}(t), \bar{R}(t))$ as $N \rightarrow \infty$:
   
   \[ \bar{S}(t) = \bar{S}(0) - \int_0^t \tilde{S}(s)\tilde{S}(s)ds, \]
   
   \[ \bar{S}(t) = \bar{I}(0)\bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t-s)\tilde{S}(s)\tilde{S}(s)ds, \]
   
   \[ \bar{I}(t) = \bar{I}(0)F^c_0(t) + \int_0^t F^c(t-s)\tilde{S}(s)\tilde{S}(s)ds, \]
   
   \[ \bar{R}(t) = \bar{I}(0)F_0(t) + \int_0^t F(t-s)\tilde{S}(s)\tilde{S}(s)ds, \]

   with $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)], \bar{\lambda}^0(t) = \mathbb{E}[\lambda^0(t)], F = \text{d.f. of } E + I, F^c = 1 - F$.
4. Equivalent to the equations in the 1927 paper of Kermack-McKendrick. We also have a CLT (with more assumptions on $\lambda$).
Varying infectivity: the LLN

- Let \( S^N(t), I^N(t), R^N(t) \) denote resp. the number of susceptible, infected and recovered indiv. in the population.
- \( S^N(t) + I^N(t) + R^N(t) = N \). Let \( \bar{S}^N = N^{-1}S^N \), same with \( I \) and \( R \).
- \((\bar{S}^N(t), \bar{I}^N(t), \bar{R}^N(t)) \rightarrow (\bar{S}(t), \bar{I}(t), \bar{R}(t)) \) as \( N \to \infty \):

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\bar{S}(t) = \bar{S}(0) - \int_0^t \bar{S}(s)\bar{I}(s)ds,
\]
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\bar{I}(t) = \bar{I}(0)\bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t-s)\bar{S}(s)\bar{I}(s)ds,
\]
\[
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\]

with \( \bar{\lambda}(t) = \mathbb{E}[\lambda(t)], \bar{\lambda}^0(t) = \mathbb{E}[\lambda^0(t)], F = \text{d.f. of } \mathcal{E} + \mathcal{I}, F^c = 1 - F \).

- Equivalent to the equations in the 1927 paper of Kermack-McKendrick. We also have a CLT (with more assumptions on \( \lambda \)).
Varying infectivity: the LLN

- Let $S^N(t), I^N(t), R^N(t)$ denote resp. the number of susceptible, infected and recovered indiv. in the population.
- $S^N(t) + I^N(t) + R^N(t) = N$. Let $\bar{S}^N = N^{-1} S^N$, same with $I$ and $R$.
- $(\bar{S}^N(t), \bar{S}^N(t), \bar{I}^N(t), \bar{R}^N(t)) \to (\bar{S}(t), \bar{S}(t), \bar{I}(t), \bar{R}(t))$ as $N \to \infty$:
  \[
  \bar{S}(t) = \bar{S}(0) - \int_0^t \bar{S}(s) \bar{S}(s)ds,
  \]
  \[
  \bar{I}(t) = \bar{I}(0) \bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t-s) \bar{S}(s) \bar{S}(s)ds,
  \]
  \[
  \bar{R}(t) = \bar{R}(0) \bar{F}_0(t) + \int_0^t \bar{F}(t-s) \bar{S}(s) \bar{S}(s)ds,
  \]
  \[
  \bar{F}(t) = \bar{F}(0) \bar{\lambda}_0(t) + \int_0^t \bar{\lambda}(t-s) \bar{S}(s) \bar{S}(s)ds,
  \]
  \[
  \text{with } \bar{\lambda}(t) = \mathbb{E}[\lambda(t)], \bar{\lambda}^0(t) = \mathbb{E}[\lambda^0(t)], \bar{F} = \text{d.f. of } \bar{\mathcal{E}} + \bar{\mathcal{I}}, \bar{F}^c = 1 - \bar{F}.
  \]
- Equivalent to the equations in the 1927 paper of Kermack - McKendrick. We also have a CLT (with more assumptions on $\lambda$).
Let us introduce the **age of infection** explicitly in our model:

\[
\mu_t^N (da) = \sum_{j=1}^{I_N(0)} 1_{\eta_j^0 > t} \delta_{\tau^N_{j,0} + t} (da) + \sum_{i=1}^{A_N(t)} 1_{\tau^N_i + \eta_i > t} \delta_{t - \tau^N_i} (da).
\]

Let \( \bar{\mu}_t^N := N^{-1} \mu_t^N \). \( \bar{\mu}^N \Rightarrow \bar{\mu} \) in \( D(\mathbb{R}^+; \mathcal{M}_F(\mathbb{R}^+)) \), where

\[
\bar{\mu}_t (da) = 1_{a \geq t} \frac{F_c(a)}{F_c(a - t)} \bar{\mu}_0 (da - t) + 1_{a < t} F_c(a) \Upsilon (t - a) da
\]

is the unique measure valued solution of the PDE

\[
\langle \partial_t \bar{\mu}_t, \varphi \rangle + \langle \partial_a \bar{\mu}_t, \varphi \rangle = \varphi(0) \Upsilon (t) - \langle \bar{\mu}_t, h \varphi \rangle,
\]

where \( F \) and \( h \) are the d.f. and the hazard rate of the duration of the infectious period, \( F_c = 1 - F \).
Let us introduce the **age of infection** explicitly in our model:

\[
\mu^N_t(da) = \sum_{j=1}^{I^N(0)} 1_{\eta_j^0 > t}\delta_{\tau_{j,0}^N+t}(da) + \sum_{i=1}^{A^N(t)} 1_{\tau_i^N + \eta_i > t}\delta_{t-\tau_i^N}(da).
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Let now \( \hat{\mu}^N_t = \sqrt{N}(\bar{\mu}^N_t - \bar{\mu}_t) \). We first note that \( \hat{\mu}^N_0(0, \cdot) \Rightarrow \hat{\mu}_0 \), where \( \hat{\mu}_0(0, a) = \bar{\mu}_0(1)^{1/2}B^0(\bar{\mu}_0([0, a])) \), \( B^0 \) is a Brownian bridge.

\( \hat{\mu}^N_t \Rightarrow \hat{\mu}_t \) in \( D(\mathbb{R}^+; H^{-1}_{loc}(\mathbb{R})) \), where \( \hat{\mu}_t \) is the (unique) solution of the SPDE

\[
\partial_t \hat{\mu}_t + \partial_a \hat{\mu}_t = -h(a)\hat{\mu}_t + \delta_0 \left[ \hat{\Upsilon}(t) + \sqrt{\Upsilon(t)} \frac{dW_{inf}(t)}{dt} \right] + \frac{\partial^2 W_{rec}(t, a)}{\partial t \partial a}.
\]
Let now $\hat{\mu}_t^N = \sqrt{N}(\bar{\mu}_t^N - \bar{\mu}_t)$. We first note that $\hat{\mu}_0^N(0, \cdot) \Rightarrow \hat{\mu}_0$, where $\hat{\mu}_0(0, a) = \bar{\mu}_0(1)^{1/2}B^0(\bar{\mu}_0([0, a]))$, $B^0$ is a Brownian bridge.

$\hat{\mu}_t^N \Rightarrow \hat{\mu}_t$ in $D(\mathbb{R}_+; H_{loc}^{-1}(\mathbb{R}))$, where $\hat{\mu}_t$ is the (unique) solution of the SPDE

$$\partial_t \hat{\mu}_t + \partial_a \hat{\mu}_t = -h(a)\hat{\mu}_t + \delta_0 \left[ \hat{\Upsilon}(t) + \sqrt{\Upsilon}(t) \frac{dW_{\inf}(t)}{dt} \right] + \frac{\partial^2 W_{rec}(t, a)}{\partial t \partial a}. $$
This is joint work with I. Drame and A. Mougabe-Peurkor.

- Suppose we are in a situation where, starting from $t = t_0$, an epidemic is declining (i.e. the mean number of susceptible individuals which an infected infects $R_{\text{eff}} < 1$), while the total number of infected individuals is $M \ll N = \text{the size of the population}$. Then deterministic models are no longer valid, the epidemic is well approximated by a sub-critical branching process, which decays essentially like an exponential $e^{-\rho(t-t_0)}$.

- The approximating branching process associated to our varying infectivity model is a non–Markov continuous time branching process. We could derive an equation for the distribution function of the time of extinction, given that a unique infected individual got infected at time $t_0$.

- If we compare the time of extinction in our model with that of a Markov SIR model with the same $R_{\text{eff}}$ and the same $\rho$, we see that the Markov model underestimates the time of extinction.
Time of extinction of an epidemic

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Kermack and McKendrick already in 1932 discussed progressive loss of immunity.

Let $s_k(t)$ be the susceptibility of the $k$–th indiv. at time $t$. The total susceptibility is $S_N(t) = \sum_{k=1}^{N} s_k(t)$. If someone gets infected at time $t$, the probability that $k$ is chosen equals $(S_N(t))^{-1} s_k(t)$.

We assume that the $i$–th infected individual after time $0$ has at time $t$ the susceptibility

$$\gamma_i(t - \tau_i^N),$$

while the $j$–th individual infected at time $0$ has at time $t$ the susceptibility

$$\gamma_j^0(t),$$

where $\gamma_i(t)$, $i \in \mathbb{N}$, are i.i.d. $[0, 1]$–valued random functions, which are non–decreasing and satisfy

$$\gamma_i(t) = 0 \text{ for } t \leq \text{ end of immunity period}.$$
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Let $s_k(t)$ be the susceptibility of the $k$–th indiv. at time $t$. The total susceptibility is $\Sigma^{N}(t) = \sum_{k=1}^{N} s_k(t)$. If someone gets infected at time $t$, the probability that $k$ is chosen equals $(\Sigma^{N}(t))^{-1} s_k(t)$.

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The functions $\lambda$ and $\gamma$

**Figure** – Illustration of the infectivity and susceptibility of an individual from the time of becoming infected, to the time of recovery, and then to time of losing immunity and becoming fully susceptible. The blue curve represents the function $\lambda(t)$ which increases to a certain value and then decreases to zero, and the orange curve represents the function $\gamma(t)$ which gradually increases to 1.
\[
\mathcal{S}^N(t) = \sum_{k=1}^{N} \gamma_k^0(t) 1_{t < \eta_{k,0}^N} + \sum_{i=1}^{A^N(t)} \gamma_i(t - \tau_{i}^N) 1_{t < \eta_{i}^N},
\]

\[
I^N(0) = \sum_{j=1}^{\lambda_j^0(t)} + \sum_{i=1}^{A^N(t)} \lambda_i(t - \tau_{i}^N),
\]

\[
A^N(t) = \int_{0}^{t} \int_{0}^{\infty} 1_{u \leq \gamma^N(s^-)} Q(ds, du), \gamma^N(t) = N^{-1} \mathcal{S}^N(t) \tilde{S}^N(t),
\]

where \(\eta_{k,0}^N\) is the time of (re)infection of the \(k\)-th individual, who is either initially susceptible (then \(\gamma_k^0(t) \equiv 1\)), or initially infected (then \(\gamma_k^0(0) = 0\)), and \(\eta_{i}^N\) is the time of reinfection of the \(i\)-th newly infected individual. \(Q\) is a standard PRM on \(\mathbb{R}_+^2\), \(\tau_{i}^N\)'s = jump times of \(A^N(t)\) i.e. the times of the “new” infections, those which happen after time 0.
Define \((\mathcal{S}^N(t), \bar{F}^N(t)) = (N^{-1}\mathcal{S}^N(t), N^{-1}\bar{F}^N(t))\).

Our main result is

**Theorem**

As \(N \to \infty\), \((\mathcal{S}^N(t), \bar{F}^N(t))\) converges in probability, locally uniformly in \(t\), to the unique solution of

\[
\mathcal{S}(t) = \mathbb{E} \left[ \gamma^0(t) \exp \left( - \int_0^t \gamma^0(r) \bar{F}(r) dr \right) \right] \\
+ \int_0^t \mathbb{E} \left[ \gamma(t-s) \exp \left( - \int_s^t \gamma(r-s) \bar{F}(r) dr \right) \right] \mathcal{S}(s) \bar{F}(s) ds,
\]

\[
\bar{F}(t) = \bar{I}(0) \lambda^0(t) + \int_0^t \lambda(t-s) \mathcal{S}(s) \bar{F}(s) ds.
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Define \((\overline{S}^N(t), \overline{F}^N(t)) = (N^{-1}S^N(t), N^{-1}F^N(t))\).

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As \(N \to \infty\), \((\overline{S}^N(t), \overline{F}^N(t))\) converges in probability, locally uniformly in \(t\), to the unique solution of

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\]

\[
\overline{F}(t) = I(0)\lambda^0(t) + \int_0^t \lambda(t-s)\overline{S}(s)\overline{F}(s)ds.
\]
We combine the exposed and infectious individuals into unique compartment of “infected individuals”. Similarly, we put the Susceptible and Recovered individuals into a unique compartment of susceptible individuals, where the recovered have susceptibility 0. The limits as $N \to \infty$ of the proportions of susceptible and infected individuals are given by the following formulas, where $F$ (resp. $F_0$) denote the d.f. of the duration of the infected period of newly infected individuals (resp. of initially infected individuals).

\[
\overline{S}(t) = \overline{S}(0) + I(0)F_0(t) - \int_0^t [1 - F(t - s)]\overline{S}(s)\overline{F}(s)ds,
\]

\[
\overline{I}(t) = \overline{I}(0)F_0^c(t) + \int_0^t F^c(t - s)\overline{S}(s)\overline{F}(s)ds.
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Only the mean of $\lambda(t)$ (and of $\lambda^0(t)$) appears in the above system of equations, while a complicated moment / exponential moment of the random functions $\gamma(t)$ (and $\gamma^0(t)$) appears in the system.
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In case of the SIR model with infection–age infectivity and recovery rate, the LLN deterministic model we obtain is exactly the model which appears in the 1927 paper of Kermack and McKendrick. We start with i.i.d. $\lambda_i(t)$’s, and we obtain the Kermack and McKendrick model with $\mathbb{E}[\lambda(t)]$.

In case of the SIRS model with infection age infectivity and recovery rate, and varying susceptibility, our model in general is quite different, since it involves not only the expectation of the random functions $\gamma(t)$. However, in case $\gamma_i(t) = \rho(t - \xi_i)$, where $\xi_i$ is the time of recovery and $\rho$ is a deterministic function (which is 0 on $\mathbb{R}_-$), then our LLN deterministic model coincides with the model of the 1932 paper of Kermack and McKendrick.
Comparison with the models of Kermack and McKendrick

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They show that, with the same mean susceptibility, progressive loss of immunity induces a more severe endemic situation, and requires a higher vaccination coverage to avoid an endemic situation.

Again, the use of simplified models may lead the authorities to decide measures which are not enough severe to be really efficient!
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G. Pang, É.P., SPDE limits for epidemic models with infection age dependent infectivity, in preparation.
HAPPY BIRTHDAY AND
LONG LIFE TO THE LMM!