Homogenization of jump Markov processes in high contrast media

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Outline

- Homogenization of high contrast elliptic problems in periodic media.
- Scaling limit of diffusion process in high contrast periodic environments.
- Nonlocal operator of convolution type. Homogenization results.
- Nonlocal convolution-type operators. Homogenization in high contrast environments.
- Scaling limit of jump processes in high contrast media. Markov (semigroup) property of the limit process in an extended space.

Classical double porosity model

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with

$$a^\varepsilon(x) = \begin{cases} \varepsilon^2, & \text{if } x \in M^\sharp \\ 1, & \text{if } x \in F^\sharp = Y \setminus M^\sharp, \end{cases}$$

here $Y = [0,1)^d$, M is a regular simply-connected open subset of $(0,1)^d$ such that $\overline{M} \subset (0,1)^d$,
$M^\sharp$ is a periodic extension of M.

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It was shown that the family of solutions u^ε converges weakly in $L^2(Q \times (0, T))$, and the limit evolution as $\varepsilon \to 0$ exhibits a memory effect. The homogenized problem reads

$$\partial_t u(x,t) = div(a_N^{\mathrm{eff}} \nabla u(x,t)) + \int_0^t D(t-s)u(s,x)ds$$

$$u|_{\partial Q} = 0, \qquad u|_{t=0} = u_0(x).$$

with an exponentially decaying kernel $D(\tau)$.

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The limit problem is well-posed.

Two-scale convergence approach

With the help of two-scale convergence method it was then shown (G. Allaire '93) that for the problem

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the following convergence result holds:

$$\int_0^T\!\!\int_Q \Big| u^\varepsilon(x,t) - \big[u_0(x,t) + \big(u_1\big(x,\frac{x}{\varepsilon},t\big)\big)\mathbf{1}_{M^\sharp}\big(\frac{x}{\varepsilon}\big)\big]\Big|^2 dxdt \to 0,$$

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where (u_0, u_1) solves the following system of equations

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$$u_0|_{\partial Q} = 0, \qquad u_0|_{t=0} = v_0(x),$$

$$\partial_t u_1(x,y,t) = \Delta_y u_1(x,y,t) - \partial_t u_0(x,t) \quad \text{in } M^\sharp,$$

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This leads to the memory effect, if we consider the first component separately.

Nonlocal operators of convolution type

We first consider moderate contrast periodic media.
Let A be an operator of the form

$$Au(x) = \int_{\mathbb{R}^d} a(x-y)\Lambda(x,y)\big(u(y)-u(x)\big)\,dy.$$

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Under the above condition A is a bounded symmetric operator in $L^2(\mathbb{R}^d)$.

Jump Markov process

The operator A is the generator of a continuous time jump Markov process $X(t)$ in $\mathbb{R}^d$ with the intensity and the jump distribution given by

$$\ell(x) = \int_{\mathbb{R}^d} a(x-y)\Lambda(x,y)\,dy, \quad \alpha(x,y) = \frac{1}{\ell(x)}a(x-y)\Lambda(x,y).$$

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This leads to the homogenization problem for convolution type operators.

Consider the scaled version of the operator A:

$$A^{\varepsilon}u(x) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\big(u(y) - u(x)\big)\,dy.$$

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Since

$$(A^\varepsilon u, u)_{L^2(\mathbb{R}^d)} = -\frac{1}{2\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)^2 dydx,$$

the equation

$$-A^\varepsilon u + \lambda u = f, \qquad f \in L^2(\mathbb{R}^d)$$

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has a unique solution u^ε for any $\lambda > 0$. Moreover $\|u^\varepsilon\|_{L^2(\mathbb{R}^d)} \leqslant \lambda^{-1}$.

Homogenization results.

In the periodic case the homogenization problem was addressed in (AP, E.Zhizhina '17).

Theorem

Let $\Lambda(x,y)$ be periodic in x and y. Then for any $f \in L^2(\mathbb{R}^d)$ the solution u^ε converges in $L^2(\mathbb{R}^d)$ as $\varepsilon \to 0$ to the solution of the problem

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As a consequence of this Theorem with some extra work we can show that for the process $X(t)$ the invariance principle holds, the covariance matrix of the limit diffusion being a^{eff}.

Nonlocal boundary value problems

The Dirichlet problem for the operator A^ε in a bounded regular domain Q reads

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Perforated domains.

Consider a periodically perforated domain $G^\varepsilon = R^d \setminus \varepsilon M^\sharp$, where $M \in (0,1)^d$ is a regular simply connected open set, $\overline{M} \subset (0,1)^d$, and $M^\sharp$ is its periodic extension.

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Let u^ε be a solution of the Neumann problem

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From now on we assume there exists $r_0 > 0$ and $a_- > 0$ such that $a(z) \geqslant a_-$ for $z \in \{z : |z| \leqslant r_0\}$.

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The following result was obtained by the variational technique

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Similar result holds for a bounded perforated domain.

High contrast media.

Let $\Lambda^\varepsilon(\xi,\eta)$ be defined by

$$\Lambda^\varepsilon(\xi,\eta) = \Lambda^0(\xi,\eta) + \varepsilon^2 p(\xi,\eta), \quad \Lambda^0(\xi,\eta) = \mathbf{1}_G(\xi)\mathbf{1}_G(\eta)$$

$$p(\xi,\eta) = p^0(\xi,\eta)\big(1 - \mathbf{1}_G(\xi)\mathbf{1}_G(\eta)\big),$$

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The high contrast operator A^ε is defined by

$$A^\varepsilon u(x) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda^\varepsilon\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)\,dy.$$

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The corresponding jump Markov process is denoted by $X_\varepsilon(t)$

Construction of the extended space E_ε

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This component $k_\varepsilon(x)$ takes values in $M^\star = \overline{M} \cup \{\star\}$, it is defined by

$$k_\varepsilon(x) \;=\; \begin{cases} \star \;, \text{ if } \; x \in \varepsilon G^\sharp, \\ \left\{\frac{x}{\varepsilon}\right\} \in \overline{M}\;, \text{ if } \; x \in \varepsilon \overline{M}^\sharp, \end{cases} \qquad \mathbb{R}^d = \varepsilon G^\sharp \cup \varepsilon \overline{M}^\sharp.$$

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Let $E_\varepsilon \subset \mathbb{R}^d \times M^\star$ be the metric space

$$E_\varepsilon = \Big\{(x, k_\varepsilon(x)),\ x \in \mathbb{R}^d,\ k_\varepsilon(x) \in M^\star\Big\},$$

with a metric that coincides with the metric in $\mathbb{R}^d$ for the first component of $(x, k_\varepsilon(x)) \in E_\varepsilon$.

From jump process on $\mathbb{R}^d$ to extended process on E_ε

Let $L_0^\infty(E_\varepsilon)$ be a Banach space of functions on E_ε vanishing as $|x| \to \infty$ with the norm

$$\|f\|_{L_0^\infty(E_\varepsilon)} = \sup_{(x,k_\varepsilon(x))\in E_\varepsilon} |f(x, k_\varepsilon(x))| = \sup_{x\in\mathbb{R}^d} |f(x, k_\varepsilon(x))|.$$

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Define the generator $\hat{L}_\varepsilon$ of the two-component random jump process $\hat{X}_\varepsilon(t) = (X_\varepsilon(t), k_\varepsilon(X_\varepsilon(t)))$ on E_ε as follows

$$(\hat{L}_\varepsilon f)(x, k_\varepsilon(x)) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} \Lambda^\varepsilon(x, y)\big(f(y, k_\varepsilon(y)) - f(x, k_\varepsilon(x))\big)\, dy,$$

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Then $\hat{L}_\varepsilon$ is the generator of strongly continuous contraction semigroup $T_\varepsilon(t)$ on $L_0^\infty(E_\varepsilon)$:

$$\|T_\varepsilon(t)f\|_{L_0^\infty(E_\varepsilon)} = \sup_{(x,k_\varepsilon(x))} |T_\varepsilon(t)f(x,k_\varepsilon(x))| \le \sup_{(x,k_\varepsilon(x))} |f(x,k_\varepsilon(x))|.$$

The generator of the limit semigroup

Letting $E = \mathbb{R}^d \times G^\star$ we denote by $C_0(E)$ the Banach space of continuous functions vanishing at infinity. A function $F = F(x,k) \in C_0(E)$ can be represented as a vector function

$$F(x,k) = \begin{pmatrix} f_0(x) \\ g(x,\xi) \end{pmatrix}$$

with $f_0(x) \in C_0(\mathbb{R}^d)$, $g(x,\xi) \in C_0(\mathbb{R}^d,\ C(\overline{M}))$.

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The norm in $C_0(E)$ is given by

$$\|F\|_{C_0(E)} = \max\Big\{ \max_{x\in\mathbb{R}^d} |f_0(x)|, \max_{x\in\mathbb{R}^d;\ \xi\in\overline{M}} |g(x,\xi)| \Big\}.$$

The generator of the limit semigroup

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$$LF(x,k) = a_N^{\mathrm{eff}} \cdot \nabla\nabla f_0(x)\mathbf{1}_{\{k=\star\}} + L_M F(x,k),$$

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where $\mathbf{1}_{\{k=\star\}}$ is the indicator function, a_N^{eff} is the effective matrix defined above, and the operator L_M is defined by

$$L_M F = \begin{pmatrix} \int\limits_M \tilde{b}(\xi)(g(x,\xi) - f_0(x))d\xi \\ \int\limits_M \tilde{a}(\xi-\xi')p^0(\xi,\xi')(g(x,\xi') - g(x,\xi))d\xi' + \tilde{c}(\xi)(f_0(x) - g(x,\xi)) \end{pmatrix}$$

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with

$$\tilde{a}(\xi) = \sum_{n\in\mathbb{Z}^d} a(\xi+n), \quad \xi \in \mathbb{T}^d,$$

$$\tilde{b}(\xi) \; = \; \frac{1}{|G|} \int\limits_G \tilde{a}(\eta-\xi)p^0(\eta,\xi)d\eta, \quad \xi \in \overline{M}, \qquad 0 < \beta_1 \le \tilde{b}(\xi) \le \beta_2,$$

$$\tilde{c}(\xi) \; = \; \int\limits_G \tilde{a}(\xi-\eta)p^0(\xi,\eta)d\eta, \quad \xi \in \overline{M}, \qquad 0 < \gamma_1 \le \tilde{c}(\xi) \le \gamma_2.$$

Matrix a_N^{eff}

Matrix $a_N^{\mathrm{eff}} = \{a_{N,ij}^{\mathrm{eff}}\}$ is defined in terms of the correctors as follows:

$$a_N^{\mathrm{eff}} = \frac{1}{2|G|} \int\limits_{G^\sharp}\int\limits_{G} a(\xi-\eta)\big[\varphi(\xi)-\varphi(\eta)-(\xi-\eta)\big]\otimes\big[\varphi(\xi)-\varphi(\eta)-(\xi-\eta)\big]\,d\eta d\xi;$$

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$$\int_G \widetilde{a}(\xi-\eta)\big(\varphi_j(\eta)-\varphi_j(\xi)\big)\,d\eta = -\int_{G^\sharp} a(\xi-\eta)(\eta-\xi)_j\,d\eta, \quad \xi \in G.$$

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We assume that φ_j are extended periodically on $G^\sharp$. Observe that the equations for different components of φ are not coupled.

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By the Hille-Yosida theorem the closure of L is the generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $C_0(E)$, that is a Feller semigroup.

The semigroup convergence $T_\varepsilon(t)F \to T(t)F$

For $F \in C_0(E)$ we define the projection operator $\pi_\varepsilon : C_0(E) \to L_0^\infty(E_\varepsilon)$ by

$$(\pi_\varepsilon F)(x, k_\varepsilon(x)) = \begin{cases} f_0(x), & \text{if } x \in \varepsilon Y^\sharp, \text{ (or } k_\varepsilon(x) = \star); \\ g(x, \{\frac{x}{\varepsilon}\}), & \text{if } x \in \varepsilon \overline{G}^\sharp, \text{ (or } k_\varepsilon(x) = \{\frac{x}{\varepsilon}\} \in \overline{G}). \end{cases}$$

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Theorem

Let $T(t)$ be the strongly continuous, positive, contraction semigroup on $C_0(E)$ with generator L, and for each $\varepsilon > 0$, $T_\varepsilon(t)$ be the strongly continuous, positive, contraction semigroup on $L_0^\infty(E_\varepsilon)$ defined above by its generators $\hat{L}_\varepsilon$.
Then for every $F \in C_0(E)$ and for all $t \geq 0$

$$\| T_\varepsilon(t)\pi_\varepsilon F - \pi_\varepsilon T(t)F \|_{L_0^\infty(E_\varepsilon)} = \| \big(T_\varepsilon(t)\pi_\varepsilon - \pi_\varepsilon T(t)\big)F \|_{L_0^\infty(E_\varepsilon)} \to 0$$

as $\varepsilon \to 0$.

Limit Markov process

Theorem

For any initial distribution $\nu \in \mathcal{P}(E)$ there exists a Markov process $\mathcal{X}(t)$ corresponding to the Feller semigroup $T(t) : C_0(E) \to C_0(E)$ with the generator L and with sample paths in $D_E[0, \infty)$.

If ν is the law of $\hat{X}_\varepsilon(0)$, then

$$\hat{X}_\varepsilon(t) \ \Rightarrow \ \mathcal{X}(t) \quad \textit{in law in} \ \ D_E[0,\infty) \ \ \textit{as} \ \ \varepsilon \to 0.$$

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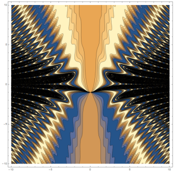
Observe that $\mathcal{X}(t) = \{x(t), k(t)\}$ is a two component process: $x(t) \in \mathbb{R}^d$, $k(t) \in G^\star$.



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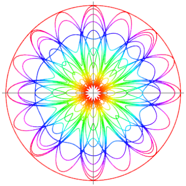

Happy Birthday to LMM !